Rectilinear Monotone $r$-Regular Planar Graphs for $r = \{3, 4, 5\}$

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Abstract

A graph $G$ consists of non-empty set of vertices (also called nodes) and the set of lines connecting two vertices called edges. The vertex set of a graph $G$ is denoted by $V(G)$ and the edge set is denoted by $E(G)$. A Rectilinear Monotone $r$-Regular Planar Graph is a simple connected graph that consists of vertices with same degree and horizontal or diagonal straight edges without vertical edges and edges crossing. This research shows that there are infinite family of rectilinear monotone $r$-regular planar graphs for $r = 3$ and $r = 4$. For $r = 5$, there are two drawings of rectilinear monotone $r$-regular planar graphs with 12 vertices and 16 vertices.

Keywords: Monotone Drawings, Planar Graphs, Rectilinear Graphs, Regular Graphs.

1. Introduction

The Pregolya River passes through a city once known as Königsberg. In the 1700s seven bridges were situated across this river in a manner similar to what you see in figure 1. The city’s residents enjoyed strolling on these bridges, but, as hard as they tried, no resident of the city was ever able to walk a route that crossed each of these bridges exactly once. The Swiss mathematician Leonhard Euler learned of this frustrating phenomenon, and in 1736 he wrote an article about it. His work on the “Königsberg Bridge Problem” is considered by many to be the beginning of the field of graph theory [1].

![Figure 1. Illustration of The Königsberg Bridge](image)

A graph $(G)$ consists of nonempty set of points called vertices (also called nodes) and the set of lines or curves connecting two points (vertices) called edges. The vertex set of a graph $G$ is denoted by $V(G)$ and the edge set is denoted by $E(G)$. Figure 2 shows a graph representation of the Königsberg Bridge. $V(G) = \{A, B, C, D\}$, representing the lands and $E(G) = \{1, 2, 3, 4, 5, 6, 7\}$, representing the bridges in Königsberg City.
A graph can be drawn in straight-line segment. If every edges of a graph drawn in straight-line segment then it is called Rectilinear Graph. The drawing of a graph without edges crossing is called Planar Graph and if it has edges crossing is called Nonplanar Graph.

A drawing of a graph is x-monotone if every edges intersects every vertical line at most once and every vertical line contains at most one vertex [2].

A graph where each of its vertex has the same degree is called Regular Graph. If degree of each of vertex is \( r \) then it called \( r \)-regular. According to [3], the number of edges in regular graphs is \( \frac{nr}{2} \).

The idea of this thesis is to construct Rectilinear Monotone \( r \)-Regular Planar Graphs for:
1. \( r = 3 \)
2. \( r = 4 \)
3. \( r = 5 \)

2. Graphs

The concepts and terminology of graphs theory are taken from [4].

**Definition 1 (Graphs)**

A simple graph \( G \) is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of \( G \) called edges. The vertex set of \( G \) is denoted \( V(G) \) and the edge set is denoted \( E(G) \).

Given two vertices, \( u \) and \( v \), of graph \( G \), the edge \( e = (u,v) \) joins \( u \) and \( v \). Common notation for the edge \( e = (u,v) \) is \( uv \). This more convenient notation is used when possible. If \( e = uv \) is an edge of \( G \), then \( u \) and \( v \) are called adjacent vertices, \( u \) and \( e \) are incident as are \( v \) and \( e \). If \( e_1 \) and \( e_2 \) are distinct edges of \( G \) with a common vertex, it is said that \( e_1 \) and \( e_2 \) are adjacent edges.

**Definition 2 (Order)**

The order of graph \( G \) is the cardinality of the vertex set of \( G \), commonly denoted \( n(G) \) or \( n \).

**Definition 3 (Size)**

The size of graph \( G \) is the cardinality of the edge set of \( G \), commonly denoted \( m(G) \) or \( m \).

**Definition 4 (Degree)**

The degree of a vertex \( v \) is the number of edges incident with \( v \), commonly denoted \( \deg v \).

In figure 3 the above terms are illustrated for two graphs. The graph in (a) has order \( n = 4 \) and size \( m = 4 \). Vertices 1 and 2 each have degree 2, vertex 3 has degree 3, and vertex 4 has degree 1. Graph (b) has order \( n = 5 \), size \( m = 10 \), and all five vertices have degree 4.
Definition 5 (Isomorphic)

A graph $G$ is isomorphic to graph $H$ if there exists a one-to-one mapping $\Phi$, called an isomorphism, from $V(G)$ onto $V(H)$ such that $\Phi$ preserves adjacency; i.e. $uv \in E(G)$ if and only if $\Phi(u) \Phi(v) \in E(H)$.

The two graphs in figure 4 (a) and (b) are isomorphic, but graph (c) is not isomorphic to either (a) or (b).

Definition 6 (Subgraphs)

A graph $G$ is subgraph of graph $H$ if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

Definition 7 (Complete Graphs)

A graph $G$ is complete if every two of its vertices are adjacent. The notation for the complete graph of order $n$ is $K_n$.

In figure 5, graph (a) is a complete graph of order six. Graph (b) is a complete graph of order 5, and graph (c) is not complete. Graphs (b) and (c) are both subgraphs of graph (a).
2.2 A Drawing of Graphs

According to [5], a drawing $D(G)$ of a graph $G$ in a plane is a mapping of the graph to the plane in such a way that vertices are mapped to distinct points of the plane and edges are mapped into arcs terminating at the appropriate vertex points. All the drawing under consideration are assumed to be simple, i.e.:
i) no edge crosses itself;
ii) adjacent edges never cross;
iii) two non-adjacent edges cross at most once;
iv) no more than two edges cross at a point of the plane.

The illegal edges for a good graph as shown in figure 6 described by [4].

Adjacent edges crossing

Nonadjacent edges crossing more than once

More than 2 edges crossing at 1 point

An edge passing through a vertex

Figure 6. Illegal Edges for A Good Graph [4]

2.3 Rectilinear Drawings

In a rectilinear (or geometric) drawing of a graph $G$, the vertices of $G$ are represented by points, and an edge joining two vertices is represented by the straight segment joining the corresponding two points. Edges are allowed to cross, but an edge cannot contain a vertex other than its endpoints [6].

Figure 7. A Rectilinear Graph [7]

2.4 Monotone Drawings

A drawing of a graph is $x$-monotone if every edges intersects every vertical line at most once and every vertical line contains at most one vertex [2].

2.5 Regular Graphs

A graph in which each vertex has the same degree is a regular graph. If each vertex has degree $r$, the graph is regular of degree $r$ or $r$-regular. Of special importance are the cubic graphs, which are regular of degree 3. Note that the null graph $N_n$ is regular of degree 0, the cycle graph $C_n$ is regular of degree 2, and the complete graph $K_n$ is regular of degree $n-1$ [8]. According to [3], the number of edges in regular graphs is $\frac{nr}{2}$. 
Note that number of edges in regular graphs is integers. It means if \( r \) is an odd number then \( n \) is an even number. Let \( r \) and \( n \) are odd numbers, \( r = 2p + 1 \) and \( n = 2q + 1 \) for \( p, q \in Z^+ \), then
\[
\frac{nr}{2} = \frac{(2q+1)(2p+1)}{2} = 2pq + p + q + \frac{1}{2} \notin Z'.
\]
Let \( n \) is an even number, \( n = 2q \) for \( q \in Z^+ \), then
\[
\frac{nr}{2} = \frac{(2q)(2p+1)}{2} = q(2p + 1) \in Z'.
\] It satisfies for \( r \) is odd number then \( n \) should be even number.

### 2.6 Planarity and Planar Embedding

A planar graph is a graph that can be drawn in the plane without crossings – that is, so that no two edges intersect geometrically except at a vertex to which both are incident. Any such drawing is a plane drawing [8].

A graph \( G \) is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. If \( G \) has no such representation, \( G \) is called nonplanar. A drawing of a planar graph \( G \) in the plane in which edges intersect only at vertices is called a planar representation (or a planar embedding) of \( G \) [1].

**Definition 8 (Planar Graphs)**

A graph is planar if it can be embedded in the plane [4].

**Definition 9 (Plane Graphs)**

A planar graph that is embedded in the plane is called a plane graph [4].

Figure 9 graph (a) is a planar graph, though as drawn it is not plane. The illustration in (b) is its plane representation.

![Figure 9](image-url)

**Note** that every subgraph of a planar graph is planar, and that every graph with a non-planar subgraph must be non-planar. It follows that any graph with \( K_{3,3} \) or \( K_5 \) as a subgraph is non-planar [8].

According to [9], a connected part of a plane which does not contain any vertices and is surrounded by edges is called a region of a planar embedding. In addition, the part outside the embedding is considered as a region, known as the exterior region (when we draw a planar graph on a plane or on a sphere, it is just like any other region). The vertices surrounding a region \( s \) are called boundary vertices and the edges surrounding \( s \) are called boundary edges. Two regions are adjacent if they share a boundary edge. Note that a region can be adjacent to itself.
In figure 10 the regions are $s_1$, $s_2$, $s_3$, $s_4$ and $s_5$ (the exterior region) and their boundary vertices and edges as well as their adjacent regions are given in the table below:

Table 1. Regions, Boundaries, and Adjacents [9]

<table>
<thead>
<tr>
<th>Region</th>
<th>Boundary Vertices</th>
<th>Boundary Edges</th>
<th>Adjacent Regions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$v_1$, $v_5$, $v_2$</td>
<td>$e_1$, $e_{10}$, $e_2$</td>
<td>$s_2$, $s_5$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$v_2$, $v_3$, $v_4$, $v_5$</td>
<td>$e_2$, $e_4$, $e_7$, $e_6$, $e_9$, $e_8$, $e_6$</td>
<td>$s_1$, $s_2$, $s_3$, $s_3$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$v_6$, $v_4$</td>
<td>$e_6$, $e_5$</td>
<td>$s_2$, $s_5$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$v_3$</td>
<td>$e_3$</td>
<td>$s_3$</td>
</tr>
<tr>
<td>$s_5$</td>
<td>$v_1$, $v_5$, $v_4$, $v_2$, $v_8$</td>
<td>$e_{10}$, $e_3$, $e_5$, $e_7$, $e_6$, $e_1$</td>
<td>$s_1$, $s_2$, $s_3$, $s_4$</td>
</tr>
</tbody>
</table>

There are theorems about planarity as the following below:

**Theorem 1 (Planarity)**

$K_{3,3}$ and $K_5$ are non-planar [8].

Figure 11 and figure 12 show that the drawings of graph $K_{3,3}$ and graph $K_5$ has 1 crossing number.
Theorem 2 (Planarity)
Every simple planar graph contains a vertex of degree at most 5 [8].

Theorem 3 (Kuratowski’s Theorem)
A graph is planar if and only if none of its subgraphs can be transformed to \(K_5\) or \(K_{3,3}\) by contracting edges [9].

Theorem 4 (Euler’s Polyhedron Formula)
If a planar embedding of a connected graph \(G\) has \(n\) vertices, \(m\) edges and \(f\) regions, then \(f + n = m + 2\) [9].

3. Research Method
The methods using in this research are study literature from books and journals. The constructions of rectilinear monotone regular planar graphs are developed by learn the characteristics of each kind of graph.

4. Results And Discussions
Rectilinear Monotone \(r\)-Regular Planar Graphs is a simple connected graph that consists of vertices with same degree and horizontal or diagonal straight edges without vertical edges and edges crossing.

4.1 Rectilinear Monotone 3-Regular Planar Graphs
Let graph \(A_n\) is a simple connected graph. There are infinite family of rectilinear monotone 3-regular planar graphs with general form:

\[A_n = \{(V_n, E_n) \mid n \in \mathbb{N} \land n \geq 3\},\]

where

\[V_n = \{p_1, p_2, \ldots, p_{n-1}, p_n, q_1, q_2, \ldots, q_{n-1}, q_n\}\]

\[E_n = \{p_kq_k | 1 \leq k \leq n\} \cup \{p_kp_{k+1} | 1 \leq k \leq n - 1\} \cup \{q_kq_{k+1} | 1 \leq k \leq n - 1\} \cup \{p_1p_n\} \cup \{q_1q_n\}\]

Figure 13 shows that the vertices of graph \(A_n\) are represented by points and every vertex has degree 3. It means graph \(A_n\) is 3-regular. Each of its edge joining two vertices is represented by straight segment and it intersects every vertical line at most a point. Then graph \(A_n\) is rectilinear and it is also \(x\)-monotone. Graph \(A_n\) is planar embedding.

To show that there are infinite family of graph \(A_n = \{(V_n, E_n) \mid n \in \mathbb{N} \land n \geq 3\}\), we must prove for \(n = 3\) there is graph \(A_3\).

4.1.1 Example of Graph \(A_3\)
\[A_3 = \{(V_3, E_3)\}\]
\[V_3 = \{p_1, p_2, p_3, q_1, q_2, q_3\}\]
\[E_3 = \{p_kq_k | 1 \leq k \leq 3\} \cup \{p_kp_{k+1} | 1 \leq k \leq 2\} \cup \{q_kq_{k+1} | 1 \leq k \leq 2\} \cup \{p_1p_3\} \cup \{q_1q_3\}\]
Then, assume that there is graph $A_n$ for all $n \in \mathbb{N}$. It will be proved that there is graph $A_{n+1}$:
1) $p_1p_n$ and $q_1q_n$ deleted $\rightarrow$ rerouted $p_n$ and $q_n$
2) $p_{n+1}q_{n+1}$ added, $p_nq_{n+1}$ and $q_nq_{n+1}$ added, $p_1p_{n+1}$ and $q_1q_{n+1}$ added

### 4.1.2 Example of Graph $A_4$

$A_4 = (V_4, E_4)$

$V_4 = \{p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4\}$

$E_4 = \{p_kq_k | 1 \leq k \leq 4\} \cup \{p_kp_{k+1} | 1 \leq k \leq 3\} \cup \{q_kq_{k+1} | 1 \leq k \leq 3\} \cup \{p_1p_4\} \cup \{q_1q_4\}$

**Figure 15. Graph $A_4$**

### 4.2 Rectilinear Monotone 4-Regular Planar Graphs

Let graph $R_n$ is a simple connected graph. There are infinite family of rectilinear monotone 4-regular planar graphs with general form:

$R_n = (V_n, E_n)$, where

$V_n = \{p_1, \ldots, p_{n-1}, p_n, q_1, \ldots, q_{n-1}, r_1, \ldots, r_{n-1}, r_n\}$

$E_n = \{p_kp_{k+1} | 1 \leq k \leq n-1\} \cup \{p_kq_k | 1 \leq k \leq n-1\} \cup \{r_kr_{k+1} | 1 \leq k \leq n-1\}$

$\cup \{r_kq_k | 1 \leq k \leq n-1\} \cup \{p_1r_1\} \cup \{p_nq_n\} \cup \{q_kq_{k+1} | 1 \leq k \leq n-1\} \cup \{q_kr_{k+1} | 1 \leq k \leq n-1\}$

**Figure 17. Graph $R_n$**

Figure 17 shows that the vertices of graph $R_n$ are represented by points and every vertex has degree 4. It means graph $R_n$ is 4-regular. Each of its edge joining two vertices is represented by straight segment and it intersects every vertical line at most one point. Then graph $R_n$ is rectilinear and it is also x-monotone. Graph $R_n$ is planar embedding.
To show that there are infinite family of graph \( R_n = \{(V_n, E_n) \mid n \in \mathbb{N} \land n \geq 3 \} \), we must prove for \( n = 3 \) there is graph \( R_3 \).

### 4.2.1 Example of Graph \( R_3 \)

\[ R_3 = \{(V_3, E_3)\} \]

\[ V_3 = \{p_1, p_2, p_3, q_1, q_2, r_1, r_2, r_3\} \]

\[ E_3 = \{p_kp_{k+1} \mid 1 \leq k \leq 2\} \cup \{p_1p_3\} \cup \{p_kq_k \mid 1 \leq k \leq 2\} \cup \{r_kr_{k+1} \mid 1 \leq k \leq 2\} \cup \{r_1r_3\} \cup \{r_kq_k \mid 1 \leq k \leq 2\} \cup \{p_1r_1\} \cup \{p_3r_3\} \cup \{q_kp_{k+1} \mid 1 \leq k \leq 2\} \cup \{q_kr_{k+1} \mid 1 \leq k \leq 2\} \]

![Graph R_3](image)

Then, assume that there is graph \( R_n \) for all \( n \in \mathbb{N} \). It will be proved that there is graph \( R_{n+1} \):

1) \( p_1p_n, r_1r_n \) and \( p_nr_n \) deleted \( \Rightarrow \) rerouted \( p_n \) and \( r_n \)
2) \( p_{n+1}r_{n+1}, q_nr_{n+1} \) and \( q_nr_{n+1} \) added, \( p_np_{n+1} \) and \( r_nr_{n+1} \) added, \( p_1p_{n+1} \) and \( r_1r_{n+1} \) added

### 4.2.2 Example of Graph \( R_4 \)

\[ R_4 = \{(V_4, E_4)\} \]

\[ V_4 = \{p_1, p_2, p_3, p_4, q_1, q_2, q_3, r_1, r_2, r_3, r_4\} \]

\[ E_4 = \{p_kp_{k+1} \mid 1 \leq k \leq 3\} \cup \{p_1p_4\} \cup \{p_kq_k \mid 1 \leq k \leq 3\} \cup \{r_kr_{k+1} \mid 1 \leq k \leq 3\} \cup \{r_1r_4\} \cup \{r_kq_k \mid 1 \leq k \leq 3\} \cup \{p_1r_1\} \cup \{p_4r_4\} \cup \{q_kp_{k+1} \mid 1 \leq k \leq 3\} \cup \{q_kr_{k+1} \mid 1 \leq k \leq 3\} \]

![Graph R_4](image)

### 4.3 Rectilinear Monotone 5-Regular Planar Graphs

This section shows two different drawings of rectilinear monotone 5-regular planar graphs. The first drawing consists of 12 vertices and 30 edges. The second consists of 16 vertices and 40 edges. Note that first drawing is irregular.

#### 4.3.1 Rectilinear Monotone 5-Regular Planar Graphs with 12 Vertices

Let graph \( W \) is a simple connected graph. There is a drawing of rectilinear monotone 5-regular planar graph with 12 vertices and 30 edges. The graph \( W = \{V, E\} \) consists of:

\[ V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\} \]

\[ E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_{11}, v_2v_3, v_2v_5, v_2v_7, v_2v_{11}, v_3v_4, v_3v_5, v_3v_6, v_4v_5, v_4v_9, v_4v_{12}, v_5v_6, v_5v_7, v_5v_8, v_5v_{10}, v_7v_8, v_7v_{10}, v_7v_{11}, v_8v_9, v_8v_{10}, v_9v_{10}, v_9v_{12}, v_{10}v_{11}, v_{10}v_{12}, v_{11}v_{12}, v_{12}v_1\} \]
4.3.2 Rectilinear Monotone 5-Regular Planar Graphs with 16 Vertices

Let graph $F$ is a simple connected graph. There is a drawing of rectilinear monotone 5-regular planar graphs with 16 vertices and 40 edges. The graph $F = \{V, E\}$ consists of:

$$
V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}
$$

$$
E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_7, v_1v_{15}, v_2v_3, v_2v_6, v_2v_{10}, v_2v_{16}, v_3v_4, v_3v_5, v_3v_6, v_4v_5, v_4v_7, v_4v_9, v_5v_6, v_5v_9, v_5v_{10}, v_5v_{11}, v_7v_8, v_7v_{11}, v_7v_{15}, v_8v_{11}, v_8v_{12}, v_9v_{12}, v_9v_{13}, v_{10}v_{13}, v_{10}v_{16}, v_{11}v_{12}, v_{11}v_{14}, v_{11}v_{15}, v_{12}v_{13}, v_{12}v_{14}, v_{13}v_{14}, v_{13}v_{16}, v_{14}v_{15}, v_{14}v_{16}, v_{15}v_{16}\}
$$

5. Conclusions And Suggestion

5.1 Conclusions

Based on the results and discussions this research has some following conclusions:

1. There are infinite family of rectilinear monotone 3-regular planar graphs with general form:

   $$
   A_n = \{(V_n, E_n) | n \in \mathbb{N} \land n \geq 3\}, \text{ where }
   $$

   $$
   V_n = \{p_1, p_2, \ldots, p_{n-1}, p_n, q_1, q_2, \ldots, q_{n-1}, q_n\}
   $$

   $$
   E_n = \{p_kq_k | 1 \leq k \leq n\} \cup \{p_kp_{k+1} | 1 \leq k \leq n-1\} \cup \{q_kq_{k+1} | 1 \leq k \leq n-1\} \cup \{p_1p_n\} \cup \{q_1q_n\}
   $$
2. There are infinite family of rectilinear monotone 4-regular planar graphs with general form:
   \[ R_n = \{ (V_n, E_n) | n \in \mathbb{N} \land n \geq 3 \} , \]
   where
   \[ V_n = \{ p_1, \ldots, p_{n-1}, p_n, q_1, \ldots, q_{n-1}, r_1, \ldots, r_{n-1}, r_n \} \]
   \[ E_n = \{ p_k p_{k+1} | 1 \leq k \leq n - 1 \} \cup \{ p_1 p_n \} \cup \{ p_k q_k | 1 \leq k \leq n - 1 \} \cup \{ r_k r_{k+1} | 1 \leq k \leq n - 1 \} \cup \{ r_1 r_n \} \cup \{ q_k r_k \} \cup \{ p_n r_n \} \cup \{ q_k p_{k+1} | 1 \leq k \leq n - 1 \} \cup \{ q_k r_{k+1} | 1 \leq k \leq n - 1 \} \]

3. There are two drawings of rectilinear monotone 5-regular planar graphs:
   a) Graph \( W \) consists of 12 vertices and 30 edges
   b) Graph \( F \) consists of 16 vertices and 40 edges

5.2 Suggestion

This thesis only shows two drawings of rectilinear monotone 5-regular planar graphs. It is suggested try to find the infinite family of rectilinear monotone 5-regular planar graphs.

6. References