K-CROSSING CRITICAL ALMOST PLANAR GRAPHS

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ABSTRACT

A graph is a pair of a non-empty set of vertices and a set of edges. Graphs can be drawn on the plane with or without crossing of its edges. Crossing number of a graph is the minimal number of crossing among all drawings of the graph on the plane. Graphs with crossing number zero are called planar. A graph is crossing critical if deleting any of its edge decreases its crossing number. A graph is called almost planar if deleting one edge makes the graph planar. This research shows graphs, given an integer $k \geq 1$, to build an infinite family of crossing critical almost planar graphs having crossing number *k*.

Keywords: Almost planar graph,crossing critical graph.

GRAF *K*-PERPOTONGAN KRITIS HAMPIR PLANAR

ABSTRAK

Sebuah graf adalah pasangan himpunan tak kosong simpul dan himpunan sisi. Graf dapat digambar pada bidang dengan atau tanpa perpotongan. Angka perpotongan adalah jumlah perpotongan terkecil di antara semua gambar graf pada bidang. Graf dengan angka perpotongan nol disebut planar. Sebuah graf dinamakan perpotongan kritis jika penghapusan sebuah sisi manapun menurunkan angka perpotongannya, sedangkan sebuah graf dinamakan hampir planar jika menghapus salah satu sisinya membuat graf yang sisa menjadi planar. Dalam penelitian ini ditunjukkan graf, yang jika diberikan bilangan bulat$k \geq 1$, dapat menghasilkan famili takhingga graf perpotongan kritis hampir planar dengan angka perpotongan *k*.

Kata kunci: Graf hampir planar, graf perpotongan kritis.

INTRODUCTION

Graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in physical, biological and social systems. Many problems of practical interest can be represented by graphs.

A graph is an abstract representation of a set of objects where some pairs of the objects are connected by links. The interconnected objects are represented by mathematical abstractions called *vertices*, and the links that connect some pairs of vertices are called *edges*. Typically, a graph is depicted in diagrammatic form as a set of dots for the vertices, joined by lines or curves for the edges. The number of edge crossings of a graph is called number of crossing.The crossing number$cr(G)$ of a graph G is the lowest number of edge crossings of a drawing of the graph G. A graph with crossing number $cr(G)=0$ is a planar graph, otherwise called a nonplanar graph.

Among many applications, the problem of crossing number very interesting and important because of its application in the optimization of chip area required in a circuit layout of Very Large Scale Integration or VLSI (Leighton, 1983 and Sherwani, 1998).Very Large Scale Integration (VLSI) is the process of creating integrated circuits by combining thousands of transistors into a single chip.The microprocessor is VLSI device. Minimizing the crossing here is also to reduce the risk of short circuit.

Structural properties of the crossing number problem studied through crossing critical graphs. A graph $G$is *k-crossing critical*if $cr(G) \geq k$ and $cr(G-e) < k$, for any edge *e* of *G*. The first construction of an infinite family KG_n of simple k-crossing-critical graphs was given by Kochol(1987).

A *nonplanar* graph *G* is *almost planar* if deleting one edge (although not any edge) makes the graph planar (Mohar, 2006). The construction family of almost planar graph was given by Pinontoan (2011).

A question is conducting the topic in this research: “Are there *k*-crossing critical almost planar graphs?”

This research is limited for *k*positive integer.

LITERATURE REVIEW

VLSI

Very Large Scale Integration (VLSI) is the process of creating integrated circuits by combining thousands of transistors into a single chip. VLSI began in the 1970s when complex semiconductor and communication technologies were being developed.

VLSI Physical Design Automation is essentially the research, development and productization of algorithms and data structures related to the physical design process. The objective is to investigate optimal arrangements of devices on a plane (or in three dimensions) and efficient interconnection schemes between these devices to obtain the desired functionality and performance. Since space on a wafer is very expensive real estate, algorithms must use the space very efficiently to lower costs and improve yield. In addition, the arrangement of devices plays a key role in determining the performance of a chip. Algorithms for physical design must also ensure that the layout generated abides by all the rules required by the fabrication process. Fabrication rules establish the tolerance limits of the fabrication process. Finally, algorithms must be efficient and should be able to handle very large designs. Efficient algorithms not only lead to fast turn-around time, but also permit designers to make iterative improvements to the layouts.

The VLSI physical design process manipulates very simple geometric objects, such as polygons and lines. As a result, physical design algorithms tend to be very intuitive in nature, and have significant overlap with graph algorithms and combinatorial optimization algorithms. In view of this observation, many consider physical design automation the study of graph theoretic and combinatorial algorithms for manipulation of geometricobjects in two and three dimensions (Sherwani, 1998).

Application of Crossing

Among many applications, the problem of crossing very interesting and important because of its application in the optimization of chip area required in a circuit layout of Very Large Scale Integration or VLSI (Leighton, 1983 and Sherwani, 1998). The microprocessor is VLSI device. Minimizing the crossing here is also to reduce the risk of short circuit.

Graph

A graph is a finite nonempty set of objects called vertices (the singular is vertex) together with a (possibility empty) set of unordered pair distinct vertices of *G* called edges.

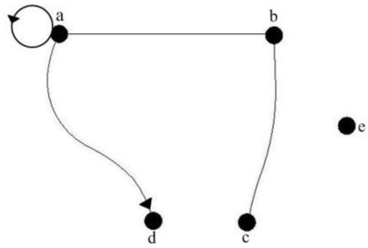


Figure 1 Example of Graph

Figure 1 provides an example of a graph on $V = \{a, b, c, d, e\}$ with $E = \{(a, a), (a, b), (a, d), (b, c)\}$.The direction of edge is indicated by placing a directed arrow on the edge, as shown here. For any edge, such as *(b,c)* we say that the edge is incident with the *vetices b, c*; *b* is said to be adjacent to *c*, where as *c* is adjacent from *b*. In addition, vertex *b* is called the origin, or source, of the edge *(b,c)* and vertex *c* is the terminus or terminating vertex. The edge *(a,a)* is an example of a loop and the vertex *e* that has no incident edge called an *isolated vertex*.

A graph *G* is called *planar* if *G* can be drawn in the plane with its edges intersecting only at vertices of *G* or a graph is planar if it can be drawn in a plane without graph edges crossing (i.e., it has graph crossing number 0) (Chartrand and Lesniak, 2000, Grimaldi, 1994).

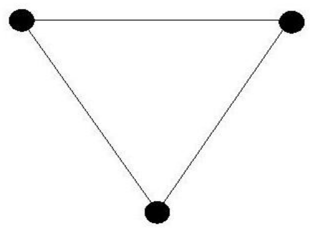


Figure 2 Planar Graph

Crossing Number

The concepts and terminology of crossing number used in this paper are taken from Cristov (2004), Ritcher and Pinontoan (2003).

The crossing number of a graph $G=(V,E)$ is $v(G)= min \{cr(D(G))\}$. Similarly, the crossing number $cr(G)$ is the minimum number of crossings over all drawing of G.

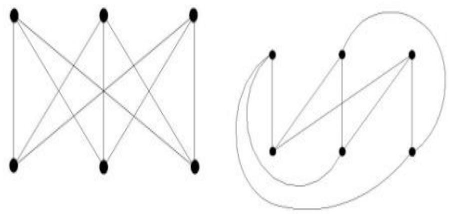
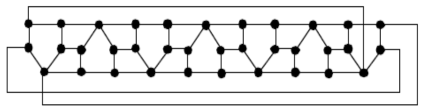


Figure 3 $K_{3,3}$

In Figure 3 Graph $K_{3,3}$, drawing $K_{3,3}$ in the plane have one crossing so that $cr(K_{3,3}) = 1$.

Crossing Critical Graph

A graph G is k-crossing critical if $cr(G) \geq k$ and $cr(G-e)<k$, for any edge e of G. It is important to study crossing critical graphs in order to understand structural properties of the crossing number problem. The only 1-crossing critical graphs are subdivisions of K_5 and $K_{3,3}$. The first construction of an infinite family KG_n of simple k-crossing critical graphs was given by Kochol(1987). KG_7 is shown in Figure 4. He showed that thisgraph have crossing numbers two. This was an infinite family of 2-crossing critical of 3-connected simple graphs.



KG_7

Figure 4
2-Crossing Critical of 3-Connected Simple Graphs

Almost Planar Graph

A graph $G$is called almost planar if deleting one edge makes the graph planar. The construction family $G_{5,8}$ of almost planar graph was given by Pinontoan (2011). $G_{5,8}$ is shown in Figure 5.

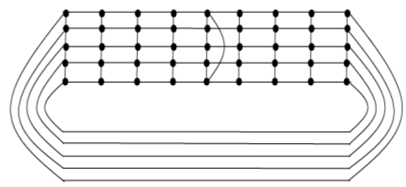


Figure 5 Family of Almost Planar Graph

Tile

The concepts and terminology of tile used in this paper are taken from Pinontoan and Richter (2003).

A *tile* is a 3-tuple $T = (G, L, R)$, where

- G is a connected graph;
- L is a finite sequence of vertices of $V(G)$, called the left-wall;
- R is a finite sequence of vertices of $V(G)$, called the right-wall; and all the vertices in L and R are distinct.

Figure 6 shows some three examples of tiles: the tiles K, M, and Z. The left-wall of each of these tiles is the sequence *abc*and the right-wall is the sequence *def* of vertices.

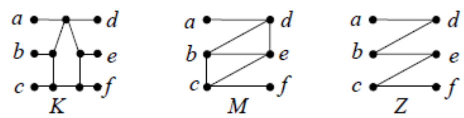


Figure 6 Tiles K, M and Z

RESEARCH METHODS

This research had use the literature study method from books and journals. From the literature study we can see, learn, showing existing result and develops it in constructing k-crossing critical almost planar graphs.

RESULTS AND DISCUSSIONS

Given an integer $k \geq 1$, there are infinite family of crossing critical almost planar graphs having crossing number k. With general form:

$G_n = \{V_n, E_n | n \geq 3\}$

$V_n = \{ u_1, u_2, u_3, \dots, u_n, v_1, v_2, v_3, \dots, v_n\}$

$E_n = \{v_k u_k | 1 \leq k \leq n\} \cup \{v_k v_{k+1} | 1 \leq k \leq n-1\} \cup \{u_k u_{k+1} | 1 \leq k \leq n-1\} \cup \{v_1 u_n\} \cup \{u_k v_{k+2} | 1 \leq k \leq n-2\}$

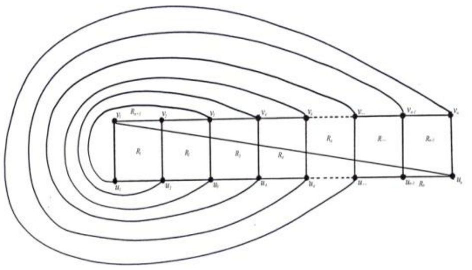


Figure 7 G_n
K-Crossing Critical Almost Planar Graph

1. To show G_n is crossing critical graph :

- $v_1 u_1$ deleted → put v_1 in R_2 region
  $v_1 u_1$ similar for $u_n v_n$
- $v_1 v_2$ deleted → put v_1 in R_n region
  $v_1 v_2$ similar for $u_{n-1} u_n$
- $v_k v_{k+1}, 2 \leq k \leq n-1$ deleted → rerouted $v_1 u_n$
- $u_k u_{k+1}, 1 \leq k \leq n-2$ deleted → rerouted $v_1 u_n$
- $u_k v_{k+2}, 1 \leq k \leq n-2$ deleted → rerouted $v_1 u_n$
- $v_k u_k$, $2 \leq k \leq n-1$ deleted → clear
- $v_1 u_n$ deleted → clear

2. To show G_n is almost planar graph:

- $v_1 u_n$ is deleted and it makes G_n as a planar graph.

Example 1-Crossing Critical Almost Planar Graphs

$G_3 = \{V_3, E_3\}$

$V_3 = \{u_1, u_2, u_3, v_1, v_2, v_3\}$

$E_3 = \{v_k u_k | 1 \leq k \leq 3\} \cup \{v_k v_{k+1} | 1 \leq k \leq 2\} \cup \{u_k u_{k+1} | 1 \leq k \leq 2\} \cup \{v_1 u_3\} \cup \{u_k v_{k+2} | k = 1\}$

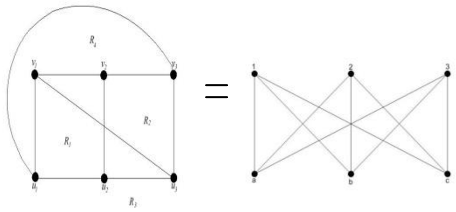


Figure 8 a)G_3 (b)$K_{3,3}$
1-Crossing Critical Almost Planar Graph

Based on figure 8 $G_3 = K_{3,3}$ because $v_1 = a$, $v_2 = 2$, $v_3 = c$, $u_1 = 1$, $u_2 =$ b, $u_3 = 3$.

We will show that G_3 is crossing critical almost planar graph.

1. To show G_3 is crossing critical graph :

- $v_1 u_1$ deleted → put v_1 in R_2 region
  $v_1 u_1$ similar for $u_3 v_3$
- $v_1 v_2$ deleted → put v_1 in R_3 region
  $v_1 v_2$ similar for $u_2 u_3$
- $v_2 v_3$ deleted → rerouted $v_1 u_3$
- $u_1 u_2$ deleted → rerouted $v_1 u_3$
- $u_1 v_3$ deleted → rerouted $v_1 u_3$
- $v_2 u_2$ deleted → clear
- $v_1 u_3$ deleted → clear

2. To show G_3 is almost planar graph:

- For 1-crossing critical almost planar graph, one of any edge of G_3 deleted makes G_3 as a planar graph.

Example 2-Crossing Critical Almost Planar Graphs

$G_4 = \{V_4, E_4\}$

$V_4 = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$

$E_4 = \{v_k u_k | 1 \leq k \leq 4\} \cup \{v_k v_{k+1} | 1 \leq k \leq 3\} \cup \{u_k u_{k+1} | 1 \leq k \leq 3\} \cup \{v_1 u_4\} \cup \{u_k v_{k+2} | 1 \leq k \leq 2\}$

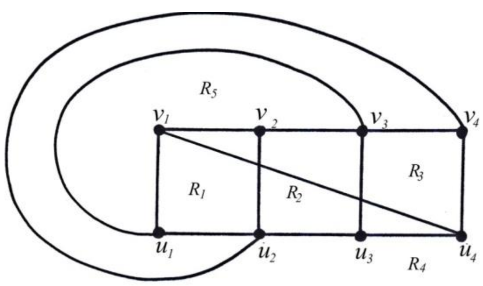


Figure 9 G_4
2-Crossing Critical Almost Planar Graph

We will show that G_4 is crossing critical almost planar graph.

1. To show G_4 is crossing critical graph :

- v_1u_1 deleted → put $v_1$in R_2 region $v_1u_1$similar for u_4v_4
- v_1v_2 deleted → put v_1 in R_4 region $v_1v_2$similar for u_3u_4
- v_2v_3 *or* $v_3v_4$deleted → rerouted v_1u_4
- u_1u_2 *or* $u_2u_3$deleted → rerouted v_1u_4
- u_1v_3 *or* u_2v_4 deleted → rerouted v_1u_4
- v_2u_2 *or* v_3u_3 deleted → clear
- v_1u_4 deleted → clear

2. To show G_4 is almost planar graph:

- $v_1u_4$is deleted and it makes G_4 as a planar graph.

Example 3-Crossing Critical Almost Planar Graphs

$G_5 = \{V_5, E_5\}$
$V_5 = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5\}$
$E_5 = \{v_ku_k | 1 \le k \le 5\} \cup \{v_kv_{k+1} | 1 \le k \le 4\} \cup \{u_ku_{k+1} | 1 \le k \le 4\} \cup \{v_1u_5\} \cup \{u_kv_{k+2} | 1 \le k \le 3\}$

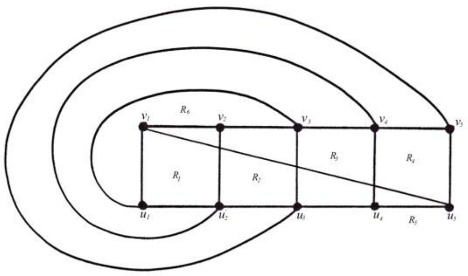


Figure 10G_5
3-Crossing Critical Almost Planar Graph

We will show that G_5 is crossing critical almost planar graph.

1. To show G_5 is crossing critical graph :

- v_1u_1 deleted → put $v_1$in R_2 region $v_1u_1$similar for u_5v_5
- v_1v_2 deleted → put v_1 in R_5 region $v_1v_2$similar for u_4u_5
- v_2v_3 *or* v_3v_4 *or* $v_4v_5$deleted → rerouted v_1u_5
- u_1u_2 *or* u_2u_3 *or* $u_3u_4$deleted → rerouted v_1u_5
- u_1v_3 *or* u_2v_4 *or* u_3v_5 deleted → rerouted v_1u_5
- v_2u_2 *or* v_3u_3 *or* v_4u_4 deleted → clear
- $v_1u_5$deleted → clear.

2. To show G_5 is almost planar graph:

- $v_1u_5$is deleted and it makes G_5 as a planar graph.

Example 4-Crossing Critical Almost Planar Graphs

$G_6 = \{V_6, E_6\}$
$V_6 = \{u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2, v_3, v_4, v_5, v_6\}$
$E_6 = \{v_ku_k | 1 \le k \le 6\} \cup \{v_kv_{k+1} | 1 \le k \le 5\} \cup \{u_ku_{k+1} | 1 \le k \le 5\} \cup \{v_1u_6\} \cup \{u_kv_{k+2} | 1 \le k \le 4\}$

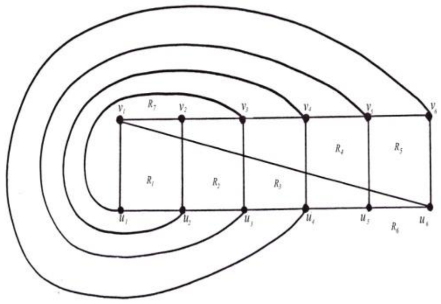


Figure 11 G_6
4-Crossing Critical Almost Planar Graph

We will show that G_6 is crossing critical almost planar graph.

1. To show G_6 is crossing critical graph :

- v_1u_1 deleted → put v_1 in R_2 region v_1u_1 similar for u_6v_6
- v_1v_2 deleted → put v_1 in R_6 region v_1v_2 similar for u_5u_6
- v_2v_3 *or* v_3v_4 *or* v_4v_5 *or* v_5v_6 deleted → rerouted v_1u_6
- u_1u_2 *or* u_2u_3 *or* u_3u_4 *or* u_4u_5 deleted → rerouted v_1u_6
- u_1v_3 *or* u_2v_4 *or* u_3v_5 *or* u_4v_6 deleted → rerouted v_1u_6
- v_2u_2 *or* v_3u_3 *or* v_4u_4 *or* v_5u_5 deleted → clear
- v_1u_6 deleted → clear

2. To show G_6 is almost planar graph:

- v_1u_6 is deleted and it makes G_6 as a planar graph.

Example 5-Crossing Critical Almost Planar Graphs

$G_7 = \{V_7, E_7\}$
$V_7 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

$E_7 = \{v_k u_k | 1 \leq k \leq 7\} \cup \{v_k v_{k+1} | 1 \leq k \leq 6\} \cup \{u_k u_{k+1} | 1 \leq k \leq 6\} \cup \{v_1 u_7\} \cup \{u_k v_{k+2} | 1 \leq k \leq 5\}$

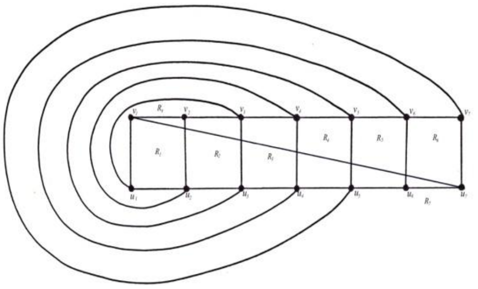


Figure 12 *G₇*
5-Crossing Critical Almost Planar Graph

We will show that G_7 is crossing critical almost planar graph.

1. To show G_7 is crossing critical graph :

- $v_1 u_1$ deleted → put v_1 in R_2 region
  $v_1 u_1$ similar for $u_7 v_7$
- $v_1 v_2$ deleted → put v_1 in R_7 region
  $v_1 v_2$ similar for $u_6 u_7$
- $v_2 v_3$ *or* $v_3 v_4$ *or* $v_4 v_5$ *or* $v_5 v_6$ *or* $v_6 v_7$ deleted → rerouted $v_1 u_7$
- $u_1 u_2$ *or* $u_2 u_3$ *or* $u_3 u_4$ *or* $u_4 u_5$ *or* $u_5 u_6$ deleted → rerouted $v_1 u_7$
- $u_1 v_3$ *or* $u_2 v_4$ *or* $u_3 v_5$ *or* $u_4 v_6$ *or* $u_5 u_7$ deleted → rerouted $v_1 u_7$
- $v_2 u_2$ *or* $v_3 u_3$ *or* $v_4 u_4$ *or* $v_5 u_5$ *or* $v_6 u_6$ deleted → clear
- $v_1 u_7$ deleted → clear

2. To show G_7 is almost planar graph:

- $v_1 u_7$ is deleted and it makes G_7 as a planar graph.

CONCLUSION

Given an integer $k \geq 1$, there exists an infinite family of crossing critical almost planar graphs having crossing number k.

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