

On Pr-essential and Pr-closed Fuzzy Submodules Characterizations via Level Submodules

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ABSTRACT

This paper investigates pure-essential and pure-closed fuzzy submodules in this setting of fuzzy module theory. The concepts are developed by incorporating purity into the classical settings of essentiality and closedness for fuzzy submodules. Several equivalent characterizations are established, and structural properties of Pr-essential and Pr-closed fuzzy submodules are investigated, particularly through their corresponding level submodules. The obtained results clarify the relationships among pure fuzzy submodules, essential fuzzy submodules, closed fuzzy submodules, and their pure-related counterparts. Examples and counterexamples are provided to illustrate the developed concepts, distinguish the resulting properties from related fuzzy notions, and demonstrate that some converses do not hold in general. These findings extend existing results in fuzzy module theory and provide a level-submodule framework for analyzing purity-related conditions in fuzzy algebra. The study also indicates directions for further investigation of the interaction between essentiality, purity, and closedness in fuzzy modules.

Keywords: Pure fuzzy submodule; Pure essential fuzzy submodule; Pure closed fuzzy submodule; Essential fuzzy submodule; Fuzzy module theory

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INTRODUCTION

Throughout the paper, $\mathcal{R}$ denotes an associative ring with identity, while all modules are taken to be unitary left $\mathcal{R}$ -modules. We first recall the fundamental notions needed in the subsequent development. Following (Rabi, 2001), a fuzzy submodule A of X is said essential (written $A \leq_e X$), whenever $A \cap B = 0_1$ for $B \leq X$, entails $B = 0_1$. In contrast $A \leq X$ is called pure fuzzy submodule, (denoted by $A \leq_{pu} X$) when it satisfies $A \cap IX = IA$ (Maysoun, 2002)). The interplay of purity and essentiality in module theory provides a structural basis for transferring these concepts to fuzzy modules. For ordinary modules, (Al Thani, 1997) developed the notion of pure-essential submodules as a refinement of essentiality. More precisely, a submodule K of an $\mathcal{R}$ -module M is called pure-essential if K is essential in every pure submodule of M containing K . Accordingly, the fuzzy analogue is defined by calling a fuzzy submodule A of X pure-essential (briefly Pr-essential), and writing $A \leq_{pe} X$, whenever every pure fuzzy submodule B of X , satisfying $A \cap B = 0_1$ must be $B = 0_1$. We also use the notion of closed fuzzy submodule ($A \leq_c X$), whenever has no proper essential enlargement of A exists in X ; equivalently, A is not property contained in any fuzzy submodule in which it is essential (Marhon & Khalaf, 2020). Motivated by this concept, the present paper introduces and studies a stronger class, namely pure-closed fuzzy submodules (briefly Pr-closed), by combining the notion of closedness with Pr-essentiality. The interaction between these two properties leads to a natural

extension of classical Pr-closed phenomena to fuzzy modules. Recent developments further demonstrate the continuing interest in essentiality, closedness and purity within fuzzy and classical module settings. (Sharma et al., 2021) investigated intuitionistic L-fuzzy essential and closed submodules and established several properties concerning these notions. In another direction, (Nimbhorkar & Khubchandani, 2022) studied fuzzy essential submodules with respect to an arbitrary fuzzy submodule, providing further characterizations and extending the framework of fuzzy essentiality. These works indicate that essential and closedness conditions in fuzzy modules theory continue to admit meaningful generalizations. The corresponding classical theory has also developed in recent years. (Ibrahim & Al-Mothafar, 2025a), studied pure-essential submodules together with pure relative complement submodules and related structure properties. More recently, (Ibrahim & Al-Mothafar, 2025b), investigated pure-closed submodules and pure-extending modules, providing new results connecting purity, closedness, and extending properties these recent contributions provide useful classical foundations for examining analogous interactions in fuzzy modules. Although these developments have advanced the study of essential, closed, and pure submodules in both fuzzy and classical settings, the interaction of these notions within fuzzy modules remains insufficiently explored. In particular, a systematic study of Pr-essential and Pr-closed fuzzy submodules, together with characterizations through their associated level submodules, is still lacking. To the best of our knowledge, the notions of Pr-essential and Pr-closed fuzzy submodules have not been systematically investigated in the fuzzy-module framework. This provides the motivation for the present work. Accordingly, this paper introduces and investigates pr-essential and pr-closed fuzzy submodules and develops several characterizations together with structural properties for these classes. Particular attention is paid to the correspondence between fuzzy conditions and their associated level submodules, providing a useful link between fuzzy-module-theoretic properties and their classical counterparts. The obtained results also clarify how pr-essentiality and pr-closedness interact with other relevant classes of fuzzy submodules and fuzzy modules. The remainder of this paper is organized as follows: **Section 2**, is devoted to Pr-essential fuzzy submodules and pure-relative complement fuzzy submodules (briefly Pr-relative), where several characterizations and related properties are established. **Section 3**, concerns Pr-closed fuzzy submodules and investigates their principal properties and relationships with other classes. Additional results and consequences are developed in the sections that follow. For clarity and terminological consistency, the terms , (fuzzy set, fuzzy submodule, and fuzzy module are abbreviated throughout the paper as fz-set, fz-submodule, and fz-module), respectively. This paper investigates pure-essential and pure-closed fuzzy submodules in this setting of fuzzy module theory. The concepts are developed by incorporating purity into the classical settings of essentiality and closedness for fuzzy submodules.

RESEARCH METHOD

This work follows a theoretical framework grounded in fuzzy module theory and level-submodule methods. The results are derived through deductive arguments and level-wise analysis.

Preliminaries

The present section establishes the theoretical groundwork for the results developed in this paper. It reviews the basic notions of fz-sets, the principal operations defined on them, and the key properties needed to support the subsequent definitions, propositions, and structural results.

Definition 1.1 (Zadeh, 1965):

Let $\mathcal{S}$ be a non-empty set, and $I = [0,1]$. A fz-set A on $\mathcal{S}$, is specified by a membership function $A: \mathcal{S} \rightarrow I$.

Definition 1.2 (Zahedi, 1992):

For $x \in \mathcal{S}$, and $t \in [0,1]$ defined the fz-set x_t , on $\mathcal{S}$ by:

$$x_t(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad y \in \mathcal{S}$$

The fz-set x_t is referred to as a *fuzzy singleton* of $\mathcal{S}$. In particular, if $x = 0$ and $t = 1$, then the corresponding fuzzy singleton is given by

$$0_1(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

This special fz-singleton is called the *zero fz-singleton*.

Definition 1.3 (Zahedi, 1992):

Let U and V be two fz-sets defined on M . comparison is interpreted pointwise as follows:

1. The fz-sets U and V are said to be equal, written $U = V$, whenever they assign the same membership value to each element of M ; that is, $U(z) = V(z)$, for every $z \in M$.
2. The fz-set U is said to be contained in V , written $U \subseteq V$, if the membership value of U does not exceed the corresponding membership value of V at any element of M . Equivalently, $U(z) \leq V(z)$, for every $z \in M$. If, in addition, there exists an element $z_0 \in M$ such that $U(z_0) < V(z_0)$, then the containment is proper. In this case, U is called a proper fz-subset of V , and we write $U \subset V$.
3. Let a_t be a fz-singleton of M . The relation $a_t \subseteq U$ holds precisely when the membership degree of a_t is dominated by that of U at every element of M ; namely, $a_t(z) \leq U(z)$, for all $z \in M$.

In particular, for if $t > 0$, this condition is equivalent to $U(a) \geq t$. Thus, the fz-singleton a_t belongs to U , or equivalently $a \in U_t$, exactly when $a_t \subseteq U$. We next introduce the fundamental operations associated with fz-sets.

Definition 1.4 (Zahedi, 1992):

Suppose that U and V are two fz-sets defined on a non-empty set X . The standard fz-union and fz-intersection of U and V are introduced pointwise as follows, for every $x \in X$:

1. The fz-union of U and V is given by: $(U \cup V)(x) = \max\{U(x), V(x)\}$.
2. The fz-intersection of U and V is given by: $(U \cap V)(x) = \min\{U(x), V(x)\}$.

More generally, if $U_i, i \in I$ constitutes a family of fz-sets on X , then

$$\left(\bigcap_{i \in I} U_i \right)(x) = \inf\{U_i(x), i \in I\}, \text{ for all } x \in X.$$

and

$$\left(\bigcup_{i \in I} U_i\right)(x) = \sup\{U_i(x), i \in I\}, \text{ for all } x \in X.$$

The following definition introduces level subsets, providing a direct connection between fz-sets and ordinary subsets.

Definition 1.5 (Martinez, 1996):

Let U be a fz-set on M . For each level $t \in [0,1]$, the corresponding t -level subset of U is defined by $U_t = \{x \in M, U(x) \geq t\}$

In other words, U_t contains precisely those elements of M whose membership values in U are not less than t . Therefore, U_t is viewed as a well-defined ordinary subset of M .

The following statement presents several fundamental properties associated with level subsets:

Remark 1.6 (Zadeh, 1965):

For two fz-subsets A and B of M . the corresponding t -level subsets, for every $t \in [0,1]$, satisfy the following identities with respect to intersection and union:

1. For every $t \in [0,1]$, $(A \cap B)_t = A_t \cap B_t$
2. $(A \cup B)_t = A_t \cup B_t$
3. A and B coincide iff $A_t = B_t$

Definition 1.7 (Zadeh, 1965):

A fz-set V on M is called the empty fz-set, written as $\emptyset$, precisely when every element of M has membership degree zero in V ; equivalently, $V(x) = 0$, for each $x \in M$.

Definition 1.8 (Zahedi, 1992):

Let M be an $\mathcal{R}$ -module, and X be a fz-set on M . We call X an fz-module (shortly $\mathcal{R}$ -fzmodule M) if its membership function is operations in the following sense:

1. $X(x-y) \geq \min\{X(x), X(y)\}$, for all $x, y \in M$.
2. $X(rx) \geq X(x)$, $\forall x \in M, r \in \mathcal{R}$.
3. $X(0) = 1$.

Definition 1.9 5 (Martinez, 1996):

Assume that χ, γ be $\mathcal{R}$ -fzmodules M . We call γ a fz- submodule of χ , if γ is pointwise contained in χ , namely, $\gamma(x) \subseteq \chi(x)$, for $x \in M$.

Proposition 1.10 (Mukhejee et al., 1996):

Let U be an fz-set on the $\mathcal{R}$ -module M , where X is an $\mathcal{R}$ -fzmodule of M . Then U is an fz-submodule of X precisely when each level subset U_t , $t \in [0,1]$, is a submodule of M .

We next establish some essential properties of fuzzy submodules, which will play a key role in the following section.

Definition 1.11 (Zahedi, 1992):

Let U, V are fz-subsets of the R -module M . Then their fz-sum $U + V$ is a fz-subset of M , where, for every $x \in M$, $(U+V)(x) = \sup \{\min\{U(u), V(v)\} : u, v \in M, x = u + v\}$

Proposition 1.12 (Zahedi, 1992):

If U and V are fz-submodules of a fz-module X , then their fz-sum $U + V$ also forms a fz-submodule of X .

Remark 1.13 (Mukhejee et al., 1996):

Suppose that χ is an $\mathcal{R}$ -fzmodule M , and let x_t be a fz-singleton contained in χ . For each fz-singleton r_k of $\mathcal{R}$, the action of r_k on x_t satisfies $r_k x_t = (rx)_\ell$, $\ell = \min\{k, t\}$.

Definition 1.14 (Rabi, 2001):

Let χ and γ be $\mathcal{R}$ -fzmodules M_1, M_2 individually. The fz-module $\chi \oplus \gamma$ over $M_1 \oplus M_2$, is determined by the membership assignment $(\chi \oplus \gamma)(a, b) = \min\{\chi(a), \gamma(b), \forall (a, b) \in M_1 \oplus M_2\}$.

When $\chi = A + B$ and $A \cap B = 0_1$, we refer to χ as internal direct sum of A and B . The corresponding decomposition is written $\chi = A \oplus B$, and both A and B are called direct summands of χ .

Definition 1.15 (Mashinchi & Zahedi, 1992):

Suppose that V is an fz-module on M . Associated with V , we introduce the following two distinguished subsets of M :

1. The support of V , denoted by V^* , is given by $V^* = \{x \in M : V(x) > 0\}$. $V^* = \cup V_t$, $t \in (0, 1]$.
2. The subset corresponding to the greatest membership value of V is defined by $V_* = V_{V(0_M)} = \{x \in M : V(x) = 1 = V(0_M)\}$.

Definition 1.16 (Maysoun, 2002):

An fz-module X is regarded as F-regular if each fz-submodule contained in X is necessarily a pure fz-submodule.

Definition 1.17 (Maysoun, 2002):

An $\mathcal{R}$ -fzmodule X on M is said to have the multiplication fz-module property if, for every fz-submodule $A \leq X$, there exists an fz-ideal K of $\mathcal{R}$ such that $A = KX$.

RESULTS AND DISCUSSION

Fundamental Properties of Pr-Essential Fz-Submodules

This section is devoted to the study of pure-essential fz-submodules, which generalize the concept of essential fz-submodules. The introduced notion combines aspects of purity and essentiality within the fz-module setting, leading to a richer structural framework. We investigate a collection of basic properties, derive several characterizations, and establish relationships between pure-essential fz-submodules and other significant classes of fz-submodules. The results below establish level-wise criteria for pr-essentiality and clarify its interaction with purity, essentiality and related structural properties of fz-submodules.

Definition 2.1:

Within a fz-module X over an $\mathcal{R}$ -module M , the pr-essentiality of an fz-submodule $A \leq X$ (denoted by $A \leq_{pr} X$), is characterized by $A \cap B = 0_1$, implies that $B = 0_1$, for every pure fz-submodule B .

Remarks and examples 2.2:

1. Each non-trivial fz-submodule of X is Pr-essential when considered as a submodule of itself.

2. Every essential fz-submodule is, in particular, Pr-essential. Thus, Pr-essential fz-submodules constitute a wider class than essential fz-submodules. However, the converse implication is not valid in general; namely, a Pr-essential fz-submodule of X may fail to be essential in X , as show by the example below.

Consider $M = Z$ with its usual Z -module structure and define $\chi: M \rightarrow [0,1]$, by $X(a) = 1$,

$$\forall a \in Z \text{ and let } A: M \rightarrow [0,1], \text{ define by : } A(a) = \begin{cases} 1 & \text{if } a \in 2Z \\ 0 & \text{otherwise} \end{cases}$$

Since $A_t = 2Z$ is a pure submodule of $\chi_t = Z$, in the ordinary module-theoretic sense according to Proposition (2.1.3) of (Maysoun, 2002), yields $A \leq_{pu} X$. To verify that A fails to be an essential fz-submodule of X , define

$$B: M \rightarrow [0,1], \text{ define by : } B(a) = \begin{cases} 1 & \text{if } a \in 3Z \\ 0 & \text{otherwise} \end{cases}$$

$$(A \cap B)(\alpha) = \begin{cases} 1 & \text{if } (\alpha) \in 6Z \\ 0 & \text{otherwise} \end{cases}$$

$(A \cap B)_* = 6Z$ is not an essential submodule of Z . proposition(2.3) of (Marhon & Khalaf, 2020) consequently shows that $A \cap B \not\leq_e X$.

3. Assume that χ is a uniform fz-module meaning that every fz-submodule of χ is essential. Then, for any fz-submodule $A \leq \chi$, we have A is essential in χ exactly when it is Pr-essential in χ .

The following proposition provides a rigorous level-submodule criterion for characterizing Pr-essential fz-submodules.

Proposition 2.3:

For an R-fz-module M represented by χ with $A \leq \chi$. The level condition $A_t \leq_{pe} X_t$, $\forall t \in [0,1]$, ensures that $A \leq_{pe} \chi$.

Proof:

To establish the assertion, let B be a pure fz-submodule of X satisfying $A \cap B = 0_1$. Accordingly, $(A \cap B)_t = (0_1)_t, \forall t \in [0,1]$. Consequently, we obtain $A_t \cap B_t = (0)$, for every $t \in (0,1]$. Moreover, A_t serves as a Pr-essential submodule of X_t , while Proposition (2.1.3) of (Maysoun, 2002) ensures that B_t is a pure submodule of X_t , for every $t \in [0,1]$. Thus, $B_t = 0$, for every $t \in [0,1]$. Hence, $B = 0_1$. Therefore, $A \leq_{pe} X$.

This results provides as level-wise sufficient condition for pr-essentiality.

The next remark records an auxiliary condition that will play a crucial role in establishing some of the forthcoming properties.

Condition (*): Let X be an $\mathcal{R}$ -fzmodule M , and let U and V be non-trivial fz-submodules of X . We assume that the inclusion $U_* \subseteq V_*$ necessarily induces the fuzzy inclusion $U \subseteq V$.

Proposition 2.4:

Consider an $\mathcal{R}$ -fzmodule X over M , satisfying the condition specified in the preceding remark, together with an fz-submodule $A \leq X$. Then A is Pr-essential in X precisely when A_* is Pr-essential in X_* ; that is, $A \leq_{pe} X$ iff $A_* \leq_{pe} X_*$.

Proof:

Suppose that $A_* \leq_{pe} X_*$. To establish that $A \leq_{pe} X$. consider a pure fz-submodule B of X satisfying $A \cap B = 0_1$. Passing to the t -level subsets, we have $(A \cap B)_t = (0)$. By Remark(1.6)(3), this relation yields $(A \cap B)_* = (0)$ and hence $A_* \cap B_* = (0)$, since $A_* \leq_{pe}$

X_* , and B_* is a pure submodule of X_* , it follows that $B_* = (0_1)_*$, Consequently, by the preceding condition(*), the inclusion $A_* \subseteq B_*$ yields $A \subseteq B$. Hence $0_1 = B \cap A = B$, and so $B = 0_1$. Consequently, A is Pr-essential fz-module of X .

Conversely, assume that $A \leq_{pe} X$. To prove that $A_* \leq_{pe} X_*$, take $N \leq_{pu} X_*$ such that $A_* \cap N = (0)$, . Associate with N the fz-submodule B of X define by :

$$B(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$$

By construction, B is an fz-submodule of X and $B_* = N$. Therefore, $A_* \cap N = A_* \cap B_* = (A \cap B)_*$. Hence $(A \cap B)_* = (0) = (0_1)_*$. By Remark (1.6)(3), give $A \cap B = 0_1$. The Pr-essentiality of A in X together with the purity of B , now implies $B = 0_1$. Consequently, $B_* = (0)$; equivalently, $N = (0)$. Hence $A_* \leq_{pe} X_*$.

Thus pr-essentiality is completely characterized at the level-submodule.

Proposition 2.5:

Let X have the PIP be a $\mathcal{R}$ -fzmodule M with $A_1 \leq B_1 \leq X$, $A_2 \leq B_2 \leq X$. Assume that $A_1 \leq_{pu} X$ and $A_2 \leq_{pu} X$. If A_1 and A_2 are Pr-essential in B_1 and B_2 (respectively), then $A_1 \cap A_2 \leq_{pe} B_1 \cap B_2$.

Proof:

Choose an arbitrary non-zero fz-submodule C of X such that $0_1 \neq C \subseteq B_1 \cap B_2 \leq_{pu} X$. Since A_1 and A_2 are Pr-essential in B_1 and B_2 , respectively, we have $A_1 \cap C \neq 0_1$ and $A_2 \cap C \neq 0_1$. On the other hand $A_1 \leq_{pu} X$ and $C \leq_{pu} X$ with X has PIP, then by proposition (3.3.7) of (Maysoun, 2002) $A_1 \cap C \leq_{pu} X$. This implies $A_1 \cap C \leq_{pu} B_2$. But $A_2 \leq_{pe} B_2$ therefore $A_1 \cap (A_2 \cap C) \neq 0_1$, which implies that $(A_1 \cap A_2) \cap C \neq 0_1$. Thus $A_1 \cap A_2 \leq_{pe} B_1 \cap B_2$.

Proposition 2.6:

Let X have the PIP a $\mathcal{R}$ -fzmodule M and A_1, A_2 are Pr-essential fz-submodules in X , then $A_1 \cap A_2 \leq_{pe} X$.

Proof:

This assertion is an immediate consequence of Proposition (2.5).

Recall that a ring is called regular, if for each $r \in \mathcal{R}$ there exists $x \in \mathcal{R}$ such that $r = rxr$ (Goodearl, 1976).

Corollary 2.7:

Let X be an $\mathcal{R}$ -fzmodule M , where $\mathcal{R}$ is a regular ring. Let $A_1 \leq B_1 \leq X$ and $A_2 \leq B_2 \leq X$. Whenever A_1 is Pr-essential in B_1 and A_2 is Pr-essential in B_2 , their intersection remains Pr-essential in the corresponding intersection; namely, $A_1 \cap A_2 \leq_{pe} B_1 \cap B_2$.

Proof:

Since $\mathcal{R}$ is a regular ring, it follows from Proposition (3.1.9) of (Maysoun, 2002). As X is F-regular, it follows that each fz-submodule of X is pure. Therefore, the assertion follows directly from Proposition (2.5). It is well known that, for submodules N and K of M such that $K \leq_{pu} M$, then k is called a Pr-relative complement of N in M whenever K is a maximal submodule with respect to the property $N \cap K = 0$, (Al Thani, 1997).

The corresponding fuzzy notion is introduced in the following definition:

Definition 2.8:

Let X be a $\mathcal{R}$ -fzmodule M , and let A be a fz-submodule of X . A pure fz-submodule B of X is said to be a Pr-relative complement of A in X whenever B is maximal, with respect to inclusion, among all pure fz-submodules of X satisfying the disjointness condition $A \cap B = 0_1$

Proposition 2.9:

Consider an fz-submodule A of X subject to the condition (*). Whenever B serves as Pr-relative complement of A in X , the direct sum $A \oplus B$ becomes a Pr-essential fz-submodule of X ; that is, $A \oplus B \leq_{pe} X$.

Proof:

The proof begins by transferring the relative-complement property to the associated level submodule namely, it suffices to show that B_* is a Pr-relative complement of A_* . The disjointness relation $B \cap A = 0_1$, together with the Remark 1.6(3) $(B \cap A)_* = (0_1)_*$, gives $B_* \cap A_* = (0)$.

Now, choose a submodule N of X_* such that $B_* \leq N$ and $N \cap A_* = (0)$. Let $C: M \rightarrow [0,1]$, defined by, $C(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$

By the way C is constructed, C is an fz-submodule of X , and moreover $C_* = N$. Consequently, $C_* \cap A_* = (0)$ which gives $(C \cap A)_* = (0_1)_*$. Hence, by the preceding remark, we obtain $C \cap A = 0_1$. Because B_* is a submodule of $C_* = N$, the preceding condition (*) transfers this inclusion to the fuzzy level yielding $B \leq C$. The maximality of B as a Pr-relative complement of A , now forces $C = B$. Remark (1.6)(3) yields $C_* = B_*$, we obtain $N = B_*$, and hence B_* is a Pr-relative complement of A_* in X_* . Accordingly, proposition(4.15) in (Al Thani, 1997) implies that $A_* \oplus B_*$ is a Pr-essential submodule of X_* . Therefore, $(A \oplus B)_*$ is Pr-essential in X_* , and hence Proposition (2.4) yields that $A \oplus B$ is a Pr-essential fz-submodule of X .

The results links pr-relative complements with pr-essentiality through direct sum.

Proposition 2.10:

For a fz-submodule $A \leq X$, pr-essentiality in X is equivalent to having a nontrivial intersection with every nonzero pure fz-submodule that is, $A \leq_{pe} X$ iff $A \cap B \neq 0_1$.

Proof:

Assume that $A \leq_{pe} X$, and let $0_1 \neq B \leq_{pu} X$. By Proposition (2.1.3) of (Maysoun, 2002), $B_* \leq_{pu} X_*$. Since $A \leq_{pe} X$, Proposition (2.4) yields that A . Consequently, $A_* \cap B_* \neq 0$, so $(A \cap B)_* \neq (0_1)_*$, then $A \cap B \neq 0_1$.

($\Leftarrow$) Let $B \leq_{pu} X$ with $A \cap B = 0_1$, we have to show that $B = 0_1$. Assume that $B \neq 0_1$ by assumption $A \cap B \neq 0_1$ which is a contradiction. Thus $A \leq_{pe} X$.

Remark 2.11

If $A \leq_{pe} B$ and $C \leq_{pe} D$, then in general $A+C \not\leq_{pe} B+D$ as show in the example below.

Example:

Consider the Z -module $M = Z \oplus (Z/2Z)$, and define the fz-subset $X: M \rightarrow [0,1]$ as follows: $X(a, b) = 1, \forall (a, b) \in Z \oplus (Z/2Z)$.

Now consider the fz-submodules $A, C: M \rightarrow [0, 1]$ and $B, D: M \rightarrow [0, 1]$, given respectively by:

$$A(a, b) = C(a, b) = \begin{cases} 1 & \text{if } (a, b) \in (2, \bar{0})Z \\ 0 & \text{otherwise} \end{cases}, \quad B(a, b) = \begin{cases} 1 & \text{if } (a, b) \in (1, \bar{0})Z \\ 0 & \text{otherwise} \end{cases}$$

$$D(a, b) = \begin{cases} 1 & \text{if } (a, b) \in (1, \bar{1})Z \\ 0 & \text{otherwise} \end{cases}$$

It is clear that:

$A_t \leq_{pe} B_t$ and $C_t \leq_{pe} D_t$, by Example (3.8) (Ahmed, 2025), so $A \leq_{pe} B$ and $C \leq_{pe} D$ by proposition (2.3).

$A \oplus C(a, b) = \sup \{ \min \{ A(a) C(b) \}, x = a+b, \text{ for all } (a, b) \in Z \oplus (Z/2Z). \}$

$$A \oplus C(a, b) = \begin{cases} 1 & \text{if } (a, b) \in (0, \bar{1})Z \\ 0 & \text{otherwise} \end{cases}$$

$(A \oplus C)_t = (0, \bar{1})Z \leq_{pu} (B \oplus D)_t$. Since $(A \oplus C)_t = A_t$, then $A_t \cap (0, \bar{1})Z = 0$, again by Example (3.8), (Ahmed, 2025), $(A \oplus C)_t \not\leq_{pe} (B \oplus D)_t$, so by proposition (2.3), $A+C \not\leq_{pe} B+D$.

Proposition 2.12:

Let $A \leq_{pe} X$ and $B \leq_{pe} X$. Assume, in addition, that the fz-sum $A + B$ is pure in X , that is, $A + B \leq_{pe} X$.

Proof:

Since $A \leq A + B \leq X$, so $A_t \leq (A + B)_t \leq X_t$ by Proposition (1.4), and since $A + B \leq_{pu} X$, then by Proposition (2.1.3) in (Maysoun, 2002), $(A + B)_t \leq_{pu} X_t$, we have $(A + B)_t \leq_{pe} X_t$ by Theorem (2.3)(b) of (Al Thani, 1997), so by Proposition (2.3). Then $A + B \leq_{pe} X$.

The next proposition investigates how Pr-essential fz-submodules behave under homomorphic images.

Proposition 2.13:

Let X and Y be two $\mathcal{R}$ -fzmodules M_1 and M_2 (respectively) and suppose that $f: X \rightarrow Y$ is a F-isomorphism. If $A \leq_{pe} X$ and every fz-submodule of X is f -invariant, then $f(A) \leq_{pe} Y$.

Proof:

Let $0_1 \neq B \leq_{pu} X$. Since f is epimorphism $f^{-1}(B) \leq_{pu} X$, by Theorem (2.3.5) (Maysoun, 2002), so $A \leq_{pe} X$, then $A \cap f^{-1}(B) \neq 0_1$. But f is monomorphism, then $f(A) \cap B \neq 0_1$. Thus $f(A) \leq_{pe} Y$.

This result shows that Pr-essentiality is preserved under F-isomorphisms under the stated invariance condition. Under certain additional conditions, the direct sum of Pr-essential fz-submodules may again be a Pr-essential fz-submodule. For this purpose, we begin with the following definition.

Definition 2.14 (Baher & Ahmed, 2024):

Let M be a non-zero module. M is called fully P-essential if every P-essential submodule is essential.

Proposition 2.15:

Let $X = X_1 \oplus X_2$ be a $\mathcal{R}$ -fzmodule M , so $M = M_1 \oplus M_2$ and X_t is fully P-essential. Suppose that $A_1 \leq X_1$ and $A_2 \leq X_2$. If $A_1 \oplus A_2 \leq_{pe} X_1 \oplus X_2$, then $A_1 \leq_{pe} X_1$ and $A_2 \leq_{pe} X_2$.

Proof:

Since X_t is fully P-essential we have $(A_1 \oplus A_2)_t \leq_e (X_1 \oplus X_2)_t$, by Proposition (5.12) of (Anderson & Fuller, 1992), $(A_1)_t \leq_e (X_1)_t$ and $(A_2)_t \leq_e (X_2)_t$, so $A_1 \leq_e X_1$ and $A_2 \leq_e X_2$ (see Proposition(2.3) in (Marhon & Khalaf, 2020)). But every essential fz-submodule is Pr-essential and we are done.

This result identifies a sufficient structural condition under which the direct sum of the given fz-submodules inherits Pr-essentiality.

Proposition 2.16:

Let $X = X_1 \oplus X_2$ be a $\mathcal{R}$ -fzmodule $M = M_1 \oplus M_2$, where for $t \in [0,1]$ $(X_1)_t$ and $(X_2)_t$ are assumed to be fully P-essential submodules of X_t . Suppose that $A_1 \leq X_1$ and $A_2 \leq X_2$. If $A_1 \leq_{pe} X_1$ and $A_2 \leq_{pe} X_2$, then it follows that $A_1 \oplus A_2 \leq_{pe} X_1 \oplus X_2$.

Proof:

Since $(X_1)_t$ and $(X_2)_t$ are fully P-essential submodules for every $t \in [0,1]$ we obtain $(A_1)_t \leq_e (X_1)_t$ and $(A_2)_t \leq_e (X_2)_t$, then by Proposition(5.20) (Anderson & Fuller, 1992), $(A_1 \oplus A_2)_t \leq_e (X_1 \oplus X_2)_t$. Accordingly, by Proposition (2.3) in (Marhon & Khalaf, 2020), $A_1 \oplus A_2 \leq_e X_1 \oplus X_2$. Since every essential fz-submodule is, in particular, Pr-essential, we finally get $A_1 \oplus A_2 \leq_{pe} X_1 \oplus X_2$.

Thus, the preceding direct-sum property can also be formulated through the corresponding level submodules.

Pr-Closed Fz-Submodules and Their Properties

This section develops the notion of Pr-closed fz-submodules in the framework of fz-modules and examines the principal properties arising from this concept. This notion may be viewed as the fuzzy analogue the classical notion pure-closed submodules. Recall that a submodule A of an $\mathcal{R}$ -module M is said to be pure-closed in M , written $A \leq_{pc} M$, when A admits no proper Pr-essential extension within M . Equivalently, every intermediate submodule B satisfying $A \leq_{pe} B \leq M$, must coincide so that $A = B$ (Al Thani, 1997). Motivated by the classical characterization, we formulate its corresponding fuzzy version and investigate several its basic properties:

Definition 3.1

An a fz-submodule A of X is called Pr-closed in X , (denoted by $A \leq_{pc} X$), if every fz-submodule B of X for which $A \leq_{pe} B \leq X$, satisfies $A = B$. We next clarify the connection between Pr-closed fz-submodules and their associated level submodules, which will be crucial for the results developed later.

Proposition 3.2:

For an $\mathcal{R}$ -fzmodule X over M satisfying the preceding condition (*), and a fz-submodule $A \leq X$ the following equivalence holds: $A \leq_{pc} X \leftrightarrow A_* \leq_{pc} X_*$.

Proof:

Assume first that $A_* \leq_{pe} N \leq X_*$. It remains to establish $A_* = N$. Define the fz-subset B by

$$: \quad B(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$$

The definition of ensures that B is an fz-submodule of X and $B_* = N$, $A_* \subseteq B_* = N$ and hence, by the preceding condition (*), $A \subseteq B$. Since $A_* \leq_{pe} N = B$, proposition (2.4), gives $A \leq_{pe} B$. As $A \leq_{pc} X$ $A = B$, so $A_* = B_* = N$, hence $A_* = N$.

Conversely, suppose that $A_* \leq_{pc} X_*$. We aim to prove that $A \leq_{pc} X$. Let $A \leq_{pe} B \leq X$, it remains to show that $A = B$. Since $A \leq_{pe} B$, Proposition (2.4) implies $A_* \leq_{pe} B_*$, it follows that $A_* = B_*$, therefore, by the preceding remark, we conclude that $A = B$.

Thus, Pr-closedness is completely characterized at the level-submodule level.

Remarks and Examples 3.3:

1. Each Pr-closed fz-submodule is a closed fz-submodule; however, the converse does not necessarily hold, as the following example demonstrates:

Example:

Work with $M = Z_{36}$ endowed with its natural Z-module structure. Take X as the fz-module of M with constant membership value 1, so that $X(a) = 1, \forall a \in Z_{36}$. $A : M \rightarrow [0,1]$, by

$$A(x) = \begin{cases} 1 & \text{if } x \in (\bar{9}) \\ 0 & \text{otherwise} \end{cases}$$

The relation $A \leq_{pc} X$, follows by exhibiting an fz-submodule B of X as given below

$$\text{Let } B : M \rightarrow [0,1] \text{ be defined by: } B(x) = \begin{cases} 1 & \text{if } x \in (\bar{4}) \\ 0 & \text{otherwise} \end{cases}$$

Since $A \oplus B = X$, it follows from Remark (3.3)(2) in (Marhon & Khalaf, 2020) that A is a direct summand. But $A_* = (\bar{9}) \not\leq_{pc} X_*$, since $A_* \leq_{pe} X_*$, so by proposition (3.2), $A \not\leq_{pc} X$.

2. Every direct summand of an fz-module X is Pr-closed in X.

Proof:

By the fact that A is a direct summand of X, one may choose an fz-submodule B of X for which the decomposition $X = A \oplus B$ holds. Hence, by the preceding remark, we have $A_* \oplus B_* = X_*$, but $A_* \oplus B_* = (A \oplus B)_*$. Moreover, by Lemma (2.3.3) of (Rabi, 2001), implies that $A_* \leq_{pc} X_*$ (see Proposition 4.11 in (Al Thani, 1997). Finally, Proposition (3.2) implies that $A \leq_{pc} X$.

3. Suppose that A is a Pr-closed fz-submodule of X. Then, for every fz-submodule B satisfying $A \subseteq B \subseteq X$, the submodule A remains Pr-closed in B; that is, $A \leq_{pc} B$.

Proof:

Let C be an fz-submodule of B such that $A \leq_{pc} C$, where A is assumed to be a Pr-closed fz-submodule of X. Because $B \leq X$, the inclusion $C \leq B$ implies that $C \leq X$. Therefore $A \leq_{pc} C \leq X$. By the assumption that A is Pr-closed in X, we immediately obtain $A = C$. Consequently, A has no proper Pr-essential extension in B. Thus, A is Pr-closed in B, and therefore $A \leq_{pc} B$.

4. pr-closedness can be characterized level-wise. In fact whenever every level submodule A_t is Pr-closed in the corresponding level submodule X_t for all $t \in [0,1]$, the fz-submodule A is Pr-closed in X.

Proof:

Assume that B is an fz-submodule of X such that $A \leq_{pe} B$. Then $A_t \leq_{pe} B_t$, for each $t \in [0,1]$. Since $A_t \leq_{pc} X_t$, for each $t \in [0,1]$, we have $A_t = B_t$, for all $t \in [0,1]$. Hence by Remark (1.6) (3), we obtain $A = B$. Thus, $A \leq_{pc} X$.

5. take X to be constant $\mathcal{R}$ -fzmodule on M , determined membership function $X(a) = 1$ for all $a \in M$. given a submodule $N < M$, define $A : M \rightarrow [0,1]$ as follows:

$$A(x) = \begin{cases} 1 & \text{if } x \in N \\ c & \text{otherwise} \end{cases}, \quad \text{where } 0 \leq c < 1. \text{ If } N \leq_{pc} M, \text{ then } A \leq_{pc} X.$$

Proof:

Fix an arbitrary $t \in [0,1]$. For $t = 0$, corresponding level subset is $A_0 = M$, if $0 < t \leq c$, then $A_t = M$. On the other hand $c < t \leq 1$, we have $A_t = N$. Thus, each level submodule A_t is a Pr-closed in $X_t = M$, for all $t \in [0,1]$ therefore, part (4), $A \leq_{pc} X$.

Proposition 3.4:

Let A and B be fz-submodules of a fz-module X satisfying $A \leq B \leq X$. If B is Pr-closed fz-submodule in X . Then the quotient fz-submodule B/A is Pr-closed in X/A ; that is, $B/A \leq_{pc} X/A$.

Proof:

Assume that $A \leq B \leq X$. Then we obtain $B_* \leq_{pe} X_*$, see Proposition (2.3). Because $B \leq_{pc} X$, Proposition (3.2), yields $B_* \leq_{pc} X_*$. Consequently, by Proposition(2.3) of (Ibrahim & Al-Mothafar, 2025), hence $B_* / A_* \leq_{pc} X_* / A_* = (B/A)_* \leq_{pc} (X/A)_*$ by Lemma (1.3.6) in (Maysoun, 2002). Thus $B/A \leq_{pc} X/A$ by proposition (3.2).

This quotient result shows that Pr-closedness is compatible with passage to factor fz-modules under the stated assumptions.

Proposition 3.5:

Let $A \leq B \leq X$, be a fz-submodules. If $A \leq_{pc} B$ and $B \leq_{pc} X$, then $A \leq_{pc} X$.

Proof:

Using Proposition (3.2), these relations transfer to the level submodules as $A_* \leq_{pc} B_*$ and $B_* \leq_{pc} X_*$. Proposition (2.7) of (Ibrahim & Al-Mothafar, 2025), then give $A_* \leq_{pc} X_*$. A final application of Proposition (3.2) once again, we conclude that $A \leq_{pc} X$.

An fz-module X is said a chained when its fz-submodules are totally ordered by inclusion; that is, for any two fz-submodules A and B of X either $A \leq B$ or $B \leq A$ (Semeein, 2010).

Corollary 3.6:

Pr-closedness is transitive along chains of fz-submodules in a chained module. Thus, for an $\mathcal{R}$ -module X over M , and fz-submodules A and B with $A \leq B \leq X$.

Proof:

The conclusion follows directly by applying Proposition (3.5) to the chain $A \leq B \leq X$.

Proposition 3.7:

Let X be a $\mathcal{R}$ -fzmodule M and let $A \leq_c X$ with X_t fully Pr-essential, then $A \leq_{pc} X$.

Proof:

Since $A \leq_c X$, then by Proposition (3.2) in (Marhon & Khalaf, 2020), $A_* \leq_c X_*$, but X_t fully Pr-essential, we have $A_* \leq_{pc} X_*$ (see Proposition (2.10) in (Ibrahim & Al-Mothafar, 2025)). Thus $A \leq_{pc} X$ by proposition (3.2).

Proposition 3.8:

For any two Pr-closed fz-submodules A and B of an fz-module X , their intersection preserves the Pr-closed property in X . Equivalently, $A \cap B \leq_{PC} X$.

Proof:

Since $A \leq_{PC} X$ and $B \leq_{PC} X$, then $A_* \leq_{PC} X_*$ and $B_* \leq_{PC} X_*$, by proposition (3.2), we have $A_* \cap B_* \leq_{PC} X_*$, then by the previous remark $(A \cap B)_* \leq_{PC} (X)_*$ (See Proposition (2.12) in (Ibrahim & Al-Mothafar, 2025)). Therefore, $A \cap B \leq_{PC} X$ by proposition (3.2).

An fz-module X is called semisimple when X admits a decomposition as a sum be expressed as the sum of simple fz-submodules (Maysoun, 2002).

Thus, the family of Pr-closed fz-submodules is stable under finite intersections.

Proposition 3.9:

Let X be an fz-module such that each fz-submodule of X is Pr-closed in X , it then follows that X is necessarily a semi-simple fz-module.

Proof:

Since every Pr-closed is a closed by Remark (2.2), we have by Remarks and Examples (2.3) in (Marhon & Khalaf, 2020), the result follows.

Remark 3.10:

Let A be a Pr-closed fz-submodule of X . Nevertheless, $(A:_R X)$ need not be Pr-closed in X , as the following example demonstrates:

Example:

Let $X : Z_{36} \rightarrow [0,1]$, define by: $X(a) = 1, \forall a \in Z_{36}$ and let $A : M \rightarrow [0,1]$, $A(x) = \begin{cases} 1 & \text{if } x \in (18) \\ 0 & \text{otherwise} \end{cases}$

Although $A_* = (18) \leq_{PC} Z_{36}$, we have $((18):_Z Z_{36})_* = 18Z \not\leq_{PC} Z$. Indeed $18Z \leq_{PC} 2Z \leq Z$ since $18Z \neq 2Z$, it follows by Proposition (3.2) that $(A:_R X)$ is not Pr-closed in X .

Corollary 3.11:

For a faithful, finitely generated multiplication $\mathcal{R}$ -fzmodule X on M , with $A \leq X$. The following conditions are equivalent:

1. A is a Pr-closed fz-submodule of X .
2. $(A:_R X)$ has the Pr-closedness property in X .
3. A can be represented in the form $A = IX$, for some $I \leq_{PC} X$.

Proof:

(1) $\Rightarrow$ (2) Assume that $A \leq_{PC} X$. By Proposition (3.2), gives $A_* \leq_{PC} X_*$. Because X is a multiplication fz-module, each X_t is likewise a multiplication fz-module as stated in Proposition (3.2.1) of (Maysoun, 2002). Therefore, Corollary (2.18) of (Ibrahim & Al-Mothafar, 2025) yields the pr-closedness $(A:_R X)_*$ in X_* . Proposition (3.2), then transfers this property to the fuzzy level giving $(A:_R X)$ is a Pr-closed in X . (2) $\Rightarrow$ (3) Since X is a multiplication, so X_t is a multiplication (see Proposition (3.2.1) in (Maysoun, 2002)), we have $A_* = (IX)_*$ for some $I_* \leq_{PC} X_*$, by Corollary (2.18) in (Ibrahim & Al-Mothafar, 2025) and by condition (*) $A = IX$.

(3) $\Rightarrow$ (1) Since $I \leq_{PC} X$, then by Proposition (3.2), $I_* \leq_{PC} X_*$, by Corollary (2.18) in (Ibrahim & Al-Mothafar, 2025a). Thus $A \leq_{PC} X$.

Open Problems for Further Investigation

i. Development of Small-Closed Fz-submodules.

Introduce a purity-based version of small-closed fz-submodules and investigate their fundamental properties, characterizations, and relationships with pure-essential and pure-closed fz-submodules.

ii. Investigation of the P-Soc (X).

Study the fz-submodule P-Soc (X), given by the common intersection of all pure-essential fz-submodules of X, and determine its structural properties, invariance conditions, and possible connections with existing fz-socles and radicals.

iii. Fuzzification of Pure-Extending Modules.

Establish a comprehensive theory of pure-extending fz-modules, obtain characterization results, and analyze the influence of purity conditions on the decomposition and closure properties of fz-modules.

CONCLUSION

In this work, we established that several fundamental properties and characterizations of pure-essential and pure-closed submodules can be naturally extended to Pr-essential and Pr-closed fz-submodules, respectively. The results obtained in this study clarify the relationships between these fuzzy notions and their corresponding level submodules, providing a systematic framework for transferring relevant classical properties to the fuzzy setting. At the same time, our analysis demonstrates that some classical results do not extend to fz-modules without additional structural assumptions. This observation emphasizes the importance of the underlying module structure in determining when such properties can be preserved in the fuzzy framework. The characterizations established throughout the paper also show the usefulness of level submodules in describing and analyzing Pr-essential and Pr-closed fz-submodules. These results contribute to the development of the theory of fz-modules and provide a basis for further investigations. In particular, future work may focus on identifying broader classes of fz-modules under which classical purity and closure properties remain valid, as well as on introducing and studying further fuzzy analogues of algebraic notions related to essentiality, purity, and closure.

REFERENCES

- Ahmed, M. A. (2025). On P-essential Submodules and P-uniform Modules. *Palestine Journal of Mathematics*, 14(2), 904–914.
- Al Thani, N. M. (1997). Pure Baer injective modules. *International Journal of Mathematics and Mathematical Sciences*, 20(3), 529–538. <https://doi.org/10.1155/S0161171297000720>
- Anderson, F.W., & Fuller, K.R. (1992). *Rings and categories of modules*, 2nd Edition. Springer.
- Baher, M. M., & Ahmed, M. A. (2024). P-Rational Submodules. *Iraqi Journal of Science*, 65(2), 878–890. <https://doi.org/10.24996/ij.s.2024.65.2.25>
- Goodearl, K. . (1976). *Ring theory: Nonsingular rings and modules*. Marcel Dekker.
- Ibrahim, O. H., & Al-Mothafar, N. S. (2025a). Pure Closed Submodules and Pure Extending Modules. *Iraqi Journal of Science*, 66(7), 2909–2916. <https://doi.org/10.24996/ij.s.2025.66.7.20>

- Ibrahim, O. H., & Al-Mothafar, N. S. (2025b). Pure Essential Submodules. *Iraqi Journal of Science*, 66(5), 1989–1995. <https://doi.org/10.24996/ij.s.2025.66.5.17>
- Marhon, H. K., & Khalaf, H. Y. (2020). Some Properties of the Essential Fuzzy and Closed Fuzzy Submodules. *Iraqi Journal of Science*, 61(4), 890–897. <https://doi.org/10.24996/ij.s.2020.61.4.22>
- Martinez, L. (1996). Fuzzy module over fuzzy rings in connection with fuzzy ideals of rings. *Journal of Fuzzy Mathematics*, 4, 843–857.
- Mashinchi, M., & Zahedi, M. M. (1992). On L-fuzzy primary submodules. *Fuzzy Sets and Systems*, 49(2), 231–236. [https://doi.org/10.1016/0165-0114\(92\)90328-2](https://doi.org/10.1016/0165-0114(92)90328-2)
- Maysoun, A. (2002). *F-regular fuzzy modules [Master's thesis]*. University of Baghdad, Baghdad.
- Mukhejee, T. ., Sen, M. ., & Roy, D. (1996). On Fuzzy Submodules and Their Radicals. *Journal of Fuzzy Mathematics*, 4, 549–557.
- Nimbhorkar, S., & Khubchandani, J. (2022). Fuzzy essential submodules with respect to an arbitrary fuzzy submodule. *TWMS Journal of Applied and Engineering*, 12(2), 435–444.
- Rabi, H.J. (2001). *Prime fuzzy submodules and prime fuzzy modules*. University of Baghdad, Baghdad.
- Semeein, S. (2010). Chained fuzzy modules. *Ibn Al-Haitham Journal for Pure and Applied Sciences*, 23(2).
- Sharma, P.K., Kanchan, & Kaur, G. (2021). Intuitionistic L-fuzzy essential and closed submodules. *Notes on Intuitionistic Fuzzy Sets*, 27(4), 44–54. <https://doi.org/10.7546/nifs.2021.27.4.44-54>
- Zadeh, L.A. (1965). Fuzzy sets. *Information and Control*, 8(3), 338–353. [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X)
- Zahedi, M.M. (1992). On L-fuzzy residual quotient modules and P-primary submodules. *Fuzzy Sets and Systems*, 51(3), 333–344. [https://doi.org/10.1016/0165-0114\(92\)90024-X](https://doi.org/10.1016/0165-0114(92)90024-X)