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Book Embedding of Infinite Family of Semi-periodic Graphs with Average Degree Between 4 and 5

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ABSTRACT

We construct an infinite family of semi-periodic graphs with average degree between 4 and 5 having pagenumber 3. We design tiles and glue these tiles, some tiles glued repeatedly, in a circular fashion, to form such an infinite family. We do book embedding of these graphs by putting the vertices on the spine of a book and the edges on 3 pages of the book without crossings, that is, with pagenumber 3.

Keywords: Book embedding; pagenumber; tile

INTRODUCTION

A graph is a mathematical structure used to model entities (vertices) and relations between the entities (edges). Formally, a graph (V, E) is a pair of a non-empty set V of vertices and a set E of edges. If u and v are vertices of a graph G, then uv is an edge of G; the vertices u and v are adjacent. We focus to undirected graphs without loops and parallel edges. If $e = uv$, then e is incident with vertex u and also with vertex v. The degree $deg(v)$ of vertex v is the number of edges incident with v. A graph G is periodic if whole G built by an identical pieces gluing in a circular fashion and it is semi-periodic if it is arranged in a circular fashion and some parts of it are built by some identical pieces. This identical piece is called a graph tile or just simply a *tile*. The concept of tile was introduced by Pinontoan & Richter (2003).

When a graph is drawn on the plane, the vertices are represented by small circles and edges by curves connecting the small circles. Two different edges can cross in a drawing. The *crossing number* $cr(G)$ of graph G is smallest possible crossings of edges of G. Pinontoan & Richter (2003) introduced a method to compute the crossing number of infinite sequences of graphs.

A *book* consists half-planes called the *pages* having a line called the *spine* as their common boundary. A book embedding of graph G is to put the vertices of G on the spine and the edges on the pages without crossing. The minimal number of pages needed to put the edges without crossing is called the *pagenumber* $\rho(G)$ of G. Book embeddings find applications in layout optimization of *very-large scale of integration* (*VLSI*) which includes microprocessor (Chung *et al*., 1987). Many studies have been conducted on book embeddings of some families of graphs, under which circulant graphs (Kainen *et al*., 2024), k-framed graphs and k-map graphs (Bekos *et al*., 2024), almost-planar crossing-critical graphs (Pinontoan *et al*., 2023), and extended periodic regular graphs (Pinontoan & Titaley, 2019). Moreover, Zheng *et al*. (2025) reviewed dynamic graph neural networks, classifying existing approaches by how they represent evolving graphs and incorporate temporal information, Depian *et al.* (2026) introduced the stack layout extension problem, and Dehghan *et al*. (2024) proposed an unsupervised framework for ranking structural node embeddings by assigning each embedding an aggregate quality score, and Alam *et al*. (2025)

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established lower and upper bounds of four and five pages, respectively, for monotone directed acyclic outerplanar graphs

Recently, Chaplick *et al*. (2026) proved that every planar graph admits a monotone topological two-page book embedding with at most (4*n*-10)/5 spine-crossing edges, Da Lozzo *et al.* (2026) showed that deciding the existence of a two-page upward book embedding is NP-complete and developed efficient algorithms for restricted cases, and Shao *et al.* (2026) determined the matching book thickness of several graph products, including corona and comb products, and investigated dispersibility under graph operations.

Richter & Thomassen (1993) found out that infinite sequence of periodic graphs having average degree 4 to 6. Book embedding of infinite sequence of crossing-critical graphs with average degree between 3.5 and 4 has been done by Pinontoan *et al.* (2019). In this research, we design tiles and construct infinite sequence of semi-periodic graphs with average degree between 4 and 5, do book embedding of these graphs with pagenumber 3.

Existing work on book embeddings has examined periodic regular graphs and crossing-critical or almost-planar graph families, while other studies establish page-number bounds for particular graph classes. The results cited above do not directly address the semi-periodic, tile-based family considered here, with average degree strictly between 4 and 5 and an exact pagenumber of 3. This study addresses that specific gap by defining a two-parameter infinite family $G_{m,n}$ through repeated and connecting tiles, establishing its average-degree range, and giving a constructive three-page embedding together with a nonplanarity-based lower bound. The contribution is therefore an explicit family and an exact embedding result, rather than a general bound for all semi-periodic graphs.

RESEARCH METHOD

The research proceeds as follows:

a. Design some tiles.
b. Glue these tiles, where some tiles are glued repeatedly, in a circular fashion, to construct an infinite family of semi-periodic graphs with average degree between 4 and 5.
c. Do book embedding for these semi-periodic graphs with average degree between 4 and 5.
d. Determine the pagenumber of these graphs.

RESULTS AND DISCUSSION

1. The tiles

We define the tiles *S*, *L*, *T*, and *R*. Each tile consists of an ordered triple, namely the graph, the left wall and the right wall. Each of the left- and right-wall is a list of vertices. For further details on tiles, we refer to the results of Pinontoan & Richter (2003).

- $S = (\{a_1, b_1, v, a_2, b_2\}, \{a_1a_2, a_1v, vb_1, va_2, vb_2, b_1b_2\}), (a_1, b_1), (a_2, b_2))$.
- $L = (\{a, b, v, c, d\}, \{ac, av, vb, vc, vd, bd, cd\}), (a, b), (c, d))$.
- $T = (\{p, c_1, d_1, u, v, q, c_2, d_2\}, \{pu, uq, c_1u, uc_2, uv, c_1d_1, c_1v, vd_1, vc_2, vd_2, c_2d_2, d_1d_2\}, (p, c_1, d_1), (q, c_2, d_2))$.
- $L = (\{c, d, v, a, b\}, \{cd, cv, cb, va, ad, dv, vb\}), (c, d), (a, b))$.

The graph of tile *S* is $(\{a_1, b_1, v, a_2, b_2\}, \{a_1a_2, a_1v, vb_1, va_2, vb_2, b_1b_2\})$, the left-wall of *S* is (a_1, b_1), and the right-wall is (a_2, b_2)).

The tiles *S*, *L*, *T*, and *R* are shown in **Fig. 1**.

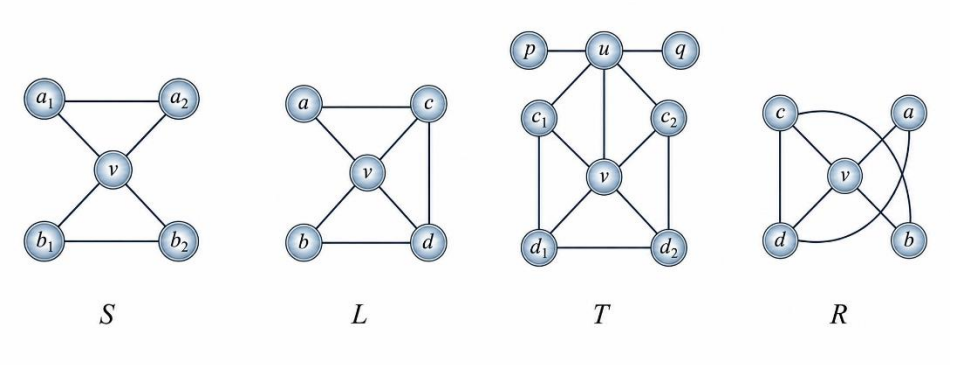


FIGURE 1. Tiles *S*, *L*, *T*, and *R*.

The gluing T_1T_2 of tile T_1 and tile T_2 are obtained by identifying the vertices of the right-wall of T_1 and the vertices of the left-wall of T_2 in the corresponding order. **Fig. 2** shows the gluing *ST* of tiles *S* and *T* from **Fig. 1**.

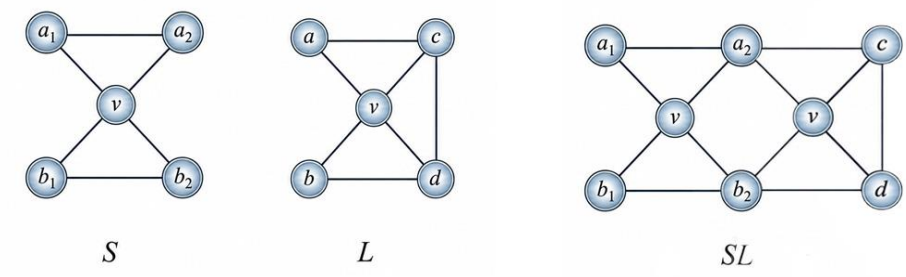


FIGURE 2. Gluing *SL* of tile *S* and *T*.

2. The Infinite Sequence of Semi-periodic Graphs with Average Degree Between 4 and 5

To construct the infinite family of semi-periodic graphs with average degree between 4 and 5, we glue the tiles as follows:

- Linear gluing S_1S_2 of two tiles *S*'s is done by identifying vertices a_2 and b_2 of S_1 with respectively vertices a_1 and b_1 of S_2.
- Linear gluing *SL* of tile *S* and tile *L* is done by identifying vertices a_2 and b_2 of tile *S* with respectively vertices *a* and *b* of tile *L*.
- Linear gluing T_1T_2 of two tiles *T*'s is done by identifying vertices *q*, c_2, and d_2 from T_1 with respectively vertices *p*, c_1, and d_1 of tile T_2.
- Linear gluing *LT* of tile *L* and tile *T* is done by identifying vertices *c* and *d* of tile *L* with respectively vertices c_1 and d_1 of tile *T*, and also identifying edge *cd* of tile *L* with edge c_1d_1 of tile *T*.
- Linear gluing *TR* of tile *T* and tile *R* is done by identifying vertices c_2 and d_2 of tile *T* with respectively vertices *c* and *d* of tile *R*, and also identifying edge c_2d_2 of tile *T* with edge *cd* of tile *R*.
- Indirect gluing T_1XT_2 of two tiles of *T*'s where other tile *X* is in between, is by identifying vertex *q* of T_1 with vertex *p* of T_2.
- Gluing *RS* of tile *R* and tile *S* is done by identifying vertices *a* and *b* of *R* with respectively vertices a_1 and b_1 of tile *S*.

The graph $G_{m,n}$ $(1 \leq m, n)$ is the circular gluing o(S^mLT^nR), where:

- S^m is the linear gluing of *m* copies of tile *S* and T^n is the linear gluing of *n* copies of tile *T*. The last copy of *S* is glued with *L* which is also glued with the first copy of *T*.

The last copy of T is then glued with R. We obtain the linear gluing S^mLT^nR. Notice that S^mLT^nR is a (big) tile.

- Circular gluing o(S^mLT^nR) is obtained by gluing R with the first S. Also indirect gluing of the last T and the first T by identifying vertex q of the last T with vertex p of the first copy of T. Notice that o(S^mLT^nR) is a graph.
- Contract all vertices with degree 2. This eliminates of the vertices p's and q's.

Furthermore, rename the vertices of graph $G_{m,n}$ ($1 \le m, n$) as follows:

- The corresponding vertices a_1, b_1, v, a_2, and b_2 of the i-th copy of S respectively, become vertices a_i, b_i, v_i, a_{i+1}, and b_{i+1}.
- The corresponding vertices a, b, v, c, and d of tile L respectively, become a_{m+1}, b_{m+1}, v_{m+1} , c_1, and d_1.
- The corresponding vertices u, c_1, d_1, v, c_2, and d_2 of the j-th copy of T respectively become vertices u_j, c_j, d_j, v_{m+j+1}, c_{j+1}, and d_{j+1}.
- The vertices c, d, v, a, and b of tile R respectively become vertices c_{n+1}, d_{m+1}, v_{m+n+2} , a_1, and b_1.

Fig. 3 shows graph $G_{2,3}$.

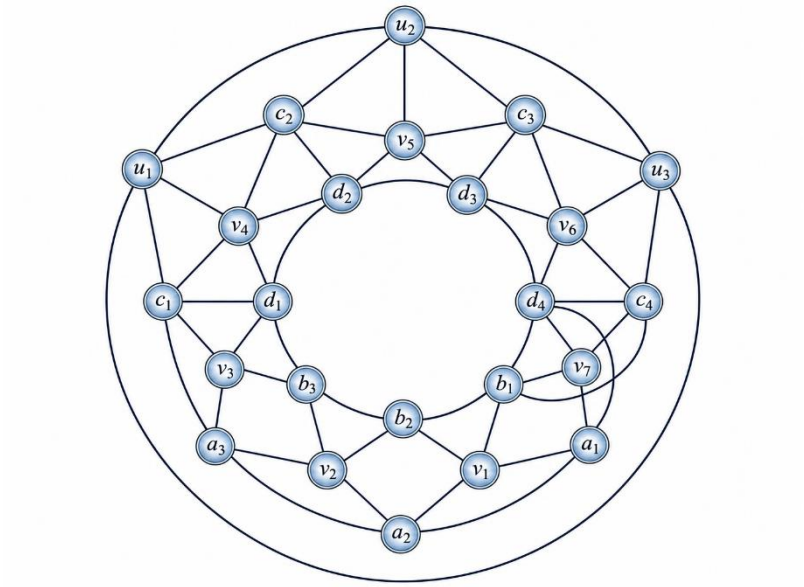


FIGURE 3. Graph $G_{2,3}$.

The average degree of $G_{m,n}$ ($1 \le m, n$) is:

$$(4(3m + 2) + 4 + 5(4n + 2) + 4) / ((3m + 2) + 1 + (4n + 2) + 1)$$
$$= (12m + 20n + 26) / (3m + 4n + 6).$$

It can be verified that:

$$4 < (12m + 20n + 26) / (3m + 4n + 6) < 5.$$

For the left inequality:

$$4(3m + 4n + 6) < 12m + 20n + 26$$
$$\Rightarrow 12m + 16n + 24 < 12m + 20n + 26$$
$$\Rightarrow 0 < 4n + 2, \text{ because } 1 \le n.$$

and for the right inequality:

$$12m + 20n + 26 < 5(3m + 4n + 6)$$
$$\Rightarrow 12m + 20n + 26 < 15m + 20n + 30.$$
$$\Rightarrow 0 < 3m + 4, \text{ because } 1 \le m.$$

3. Book Embedding of Graphs $G_{m,n}$

For $1 \le m, n$, graph $G_{m,n}$ contains a subgraph that is a subdivision of $K_{3,3}$ and hence $G_{m,n}$ is a non-planar graph. Therefore, the pagenumber of the graph $G_{m,n}$ is $\rho(G_{m,n}) \ge 3$.

Now, we show that $G_{m,n}$ can be book-embedded in 3 pages, which shows that $\rho(G_{m,n}) \le 3$. This will conclude that $\rho(G_{m,n}) = 3$. The book embedding is as follows:

- Put the vertices of $G_{m,n}$ on the spine in the following order: $a_1, b_1, v_1, a_2, b_2, v_2, \ldots, a_{m+1}, b_{m+1}, v_{m+1}, c_1, d_1, v_{m+2}, u_1, c_2, d_2, v_{m+3}, u_2, \ldots, c_n, d_n, v_{m+n+1}, u_n, c_{n+1}, d_{n+1}$, and v_{m+n+2}.
- Put the following edges on page 1: a_iv_i $(1 \le i \le m+1)$ and b_ib_{i+1}, v_ib_{i+1} $(1 \le i \le m)$ as well d_id_{i+1}, $v_{m+1+i}d_i$, $v_{m+1+i}c_i$, $v_{m+1+i}u_i$ $(1 \le i \le n)$, and also u_ic_{i+1} $(1 \le i \le n)$, $c_{i+1}d_{i+1}$ $(1 \le i \le n+1)$, $b_{m+1}d_1$, $b_{m+1}d_1$, a_1v_{m+n+2}, a_1d_{n+1}, and $d_{n+1}v_{m+n+2}$,
- Put the following edges on page 2:
  a_iv_i $(1 \le i \le m+1)$, and a_ia_{i+1}, v_ia_{i+1} $(1 \le i \le m)$, as well c_iv_{i+m+1}, d_iv_{i+m+1}, c_iu_i $(1 \le i \le n)$, and also u_iu_{i+1} $(1 \le i \le n\text{-}1)$, c_1u_1, u_1u_n, $c_{n+1}v_{m+n+2}$, u_nc_{n+1}, and $c_{n+1}d_{n+1}$.
- Finally, put the following edges on page 3:
  b_1v_{m+n+2} and b_1c_{n+1}.

Fig. 4 shows a book embedding of $G_{2,3}$ shown in **Fig. 3**. The vertices are put on the spine with the following order:

$a_1, b_1, v_1, a_2, b_2, v_2, a_3, b_3, v_3, c_1, d_1, v_4, u_1, c_2, d_2, v_5, u_2, c_3, d_3, v_6, u_3, c_4, d_4, v_7$.

The edges on page 1 are drawn above the spine with solid lines; those on page 2 are drawn below the spine with solid lines; and those on page 3 are drawn with dotted lines.

The construction supplies infinitely many nonplanar examples for which the pagenumber is known exactly, rather than only bounded from above. Its two independent parameters control the numbers of repeated S and T tiles, allowing the family to vary in size and average degree while retaining a three-page embedding. The tile-by-tile description and explicit spine order provide a reproducible way to generate examples and verify page assignments. Consequently, the family can serve as a structured set of instances for studying how repeated local graph structure influences book embeddings and for testing layout procedures on nonplanar graphs. These are theoretical and methodological benefits; the paper does not establish a specific improvement in practical VLSI performance.

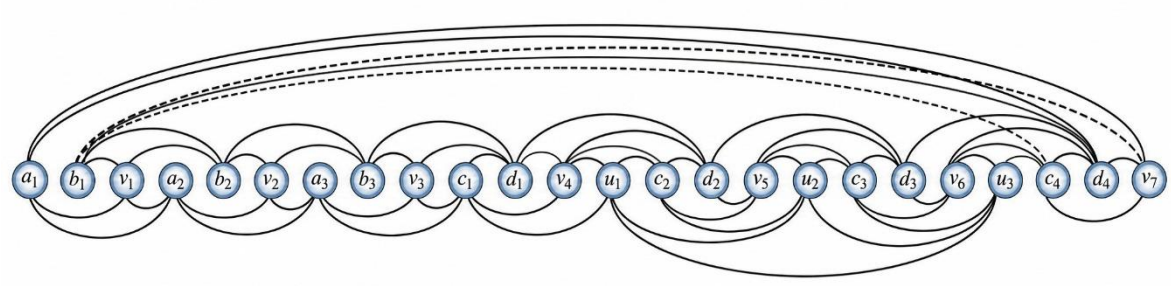


FIGURE 4. Book embedding of graph $G_{2,3}$.

CONCLUSION

There is an infinite family $G_{m,n}$ $(1 \le m, n)$ of semi-periodic graphs with average degree between 4 and 5 having pagenumber $\rho(G_{m,n}) = 3$.

Author Contributions

Benny Pinontoan: Conceptualization and methodology. **Jullia Titaley**: Literature review and formal analysis. **Chriestie E.J.C. Montolalu**: Writing-original draft, review, and editing. All authors reviewed and approved the final manuscript and agreed to be accountable for its content and integrity.

Declaration of AI Tool Usage

During the preparation of this manuscript, the authors used ChatGPT (OpenAI) to assist with language editing and grammar correction. All AI-assisted content was carefully reviewed and edited by the authors to ensure accuracy, clarity, and academic integrity. The authors take full responsibility for the content and integrity of the final manuscript.

Conflict of Interest

The authors declare no conflicts of interest related to this study.

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