Stability and Behaviour of Rotationally Symmetric Harmonic Maps From A Ball Into A Sphere

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Abstract

Rotationally Symmetric Harmonic Maps from a Ball into a Sphere has been studied before. The systems conducted in this study can be analyzed further by checking its stability and its behavior in the system. This paper will show how to determine the stability of the system and its behaviour by reducing it into a damped pendulum equation differential equation.

Keywords: Rotational symmetry, Harmonic maps, Stability, Damped pendulum equation.

Kestabilan dan Sifat dari Pemetaan Harmonik yang Berotasi Simetris dari Sebuah Bola ke Sebuah Sphere

Abstrak

Pemetaan Harmonik yang Berotasi Simmetris dari sebuah bola ke sebuah "sphere" telah dipelajari sebelumnya. Sistem yang ditemukan dalam studi ini bisa dianalisa lebih lanjut dengan memeriksa kestabilan dan sifat-sifatnya dalam sistem. Tulisan ini akan menunjukkan cara untuk menentukan kestabilan sistem dan sifat-sifatnya dengan mengubahnya ke dalam bentuk persamaan differensial pendulum sederhana.

Kata kunci:Rotasi simetris, Pemetaan harmonic, Kestabilan, Persamaan pendulum sederhana.

1. Introduction

An object with ***rotational symmetry*** is an object that looks the same after a certain amount of rotation. A map between two compact Riemannian manifolds is a ***harmonic map*** if it is a critical point for the energy functional. For example, a map from a circle to the equator of standard 2-sphere is a harmonic map, and so are the maps that take the circle and map it around the equator n times, for any integer n.

Let M be a n-dimensional Riemannian manifold (with or without boundary) with a smooth Riemannian metric g. In a local coordinates around fixed point $p \in M$, g can be represented by

$$g = g_{ij} dx_i \otimes dx_j,$$

where g_{ij} is a positive definite symmetric $n \times n$ matrix. Let $(g^{ij}) = (g_{ij})^{-1}$ be the inverse matrix of (g_{ij}) and the volume element of $(M; g)$ is

$$dv_g = \sqrt{|g|}dx,$$

where $|g| = \det(g_{ij})$. Let N be another l-dimensional compact Riemmanian manifold (without boundary) with a smooth Riemmanian metric h.

For a map $u: M \to N$, its Dirichlet energy functional is defined by

$$E(u) = \int_M e(u)\, dv_g$$

where the density function $e(u)$ is given by

$$e(u)(x) = \tfrac{1}{2}|\nabla u(x)|^2 = \tfrac{1}{2} \sum_{\alpha,\beta,i,j} g^{ij}(x) h_{\alpha\beta}\big(u(x)\big) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}$$

A smooth map u from M to N is said to be a harmonic map if u is a critical point of the Dirichlet energy functional E; i.e. it satisfies

$$\Delta_M u + A(u)(\nabla u, \nabla u) = 0$$

in M, where Δ_M is the Laplacian operator with respect to the Riemmanian metric of M and A is the second fundamental form of M.

In this case, the harmonic map was considered to be from unit ball to unit sphere which satisfies a variational problem in Euler equation form:

$$-\Delta u = |\nabla u|^2 . u. \tag{1.1}$$

Let $\kappa \geq 0$ be an upper bound for the sectional curvature of N and $K_p(q)$ the open geodesic ball in N with center q and radius ρ. Assuming essentially the size restriction

$$f(\partial M) \subset K_p(q),\ \ \rho \leq \frac{\pi}{2\sqrt{\kappa}}, \tag{1.2}$$

Hilderbrandt et. Al. [5] showed existence of a "small" smooth harmonic maps satisfying (1.2). This was shown by considering solution of a Dirichlet problem, which is smooth in the interior and minimizes the energy in the class in $K_p(q)$ having the boundary values f. In the case where N is the standard sphere the smallness condition restricts the image of the boundary values and the solution to an open half-sphere.

Suppose B^n denotes the compact unit ball in the Euclidean space R^n, and S^n the unit sphere in R^{n+1}. Select the point given by the $(n + 1)$-th standard base vector e_{n+1} in R^{n+1} as northpole of the sphere. Every map $u: B^n \to S^n$ can be written in the form

$$u(x) = (g(x). \sin\varphi(x), \cos\varphi(x)) \tag{1.3}$$

with maps

$$\varphi: B^n \to [0, \pi],\ \ g: B^n \to S^{n-1} \subset R^n$$

φ measures the Riemannian distance from $u(x)$ to the northpole on the sphere and is called *radius function* of u. The map g is uniquely defined by u except for points x where $u(x) = \pm e_{n+1}$. A map $u: B^n \to S^n$ is *rotationally symmetric* if and only if

$$g(x) = \frac{x}{|x|} \text{ and } \varphi(x) = \Phi(|x|). \tag{1.4}$$

In order to study the question what happens if the smallness condition is violated, the family of boundary values can be defined as

$$f_\rho: \partial B^n \to S^n, f_\rho(x) = (x. \sin\rho, \cos\rho)$$

which is considered as an homotopy with parameter $\rho \in [0, \pi]$. For $n = 2$ as shown by Lemaire in [3] and $n \geq 3$ by Wood in [1], harmonic maps from B^n into a Riemannian manifold are constant if they are constant on the boundary. Therefore, for $\rho = 0$ and $\rho = \pi$ the Dirichlet problem has exactly one trivial solution.

In this project, the harmonic maps were restricted to rotationally symmetric behavior. The stability of the result will be shown by reducing the discussion to ordinary differential equation, especially to the damped pendulum equation.

2. Rotationally Symmetric Harmonic Maps

We are going to study rotationally symmetric maps with finite energy. Recall that every map $u: B^n \to S^n$ can be written in the form

$$u(x) = (g(x). \sin\varphi(x), \cos\varphi(x)) \tag{2.1}$$

with maps

$$\varphi: B^n \to [0, \pi],\ \ g: B^n \to S^{n-1} \subset R^n$$

φ measures the Riemannian distance from $u(x)$ to the northpole on the sphere and is called *radius function* of u. A map $u: B^n \to S^n$ is rotationally symmetric if and only if

$$g(x) = \frac{x}{|x|} \text{ and } \varphi(x) = \Phi(|x|). \tag{2.2}$$

In this case, metric form of the manifold can be written as [2]:

$$|u|^2 = |\varphi|^2 + f^2(\varphi). |g|^2$$

Since the projection assumed is projection from unit ball to unit sphere, then this can be assumed as a function in geodesic coordinate of a sphere, i.e. $f(\varphi) = \sin\varphi$.

Therefore, for $u(x) = \left(\frac{x}{r}\sin\Phi(r), \cos\Phi(r)\right)$, with radius function $\Phi: [0,1] \to [0, \pi]$ depending only on $r = |x|$, the metric form can be written as

$$|u|^2 = |\Phi(r)|^2 + sin^2(\Phi(r)).\left|\frac{x}{r}\right|^2$$

And

$$|\nabla u|^2 = |\nabla\Phi(r)|^2 + sin^2(\Phi(r)).\left|\nabla\left(\frac{x}{r}\right)\right|^2$$

Suppose $\Psi(t) = \Phi(e^t)$, $\Psi: (-\infty, 0] \to [0, \pi]$
$\Psi'(t) = e^t\Phi'(e^t)$
$\Psi''(t) = e^t\Phi'(e^t) + e^{2t}\Phi''(e^t)$
and can be written in differential equation [2,4] as follows:

$$\Psi''(t) + (n-2)\Psi'(t) - \frac{n-1}{2}\sin 2\Psi(t) = 0 \quad (2.3)$$

The energy can be expressed in term of $\Psi(t)$:

$$E(u) = \frac{\omega_n}{2}\int_{-\infty}^{0}\left[\Phi'^2(e^t) + \frac{n-1}{e^{2t}}sin^2\Phi(e^t)\right]e^{(n-1)t}e^t dt$$

$$= \frac{\omega_n}{2}\int_{-\infty}^{0}[e^{2t}\Phi'^2(e^t) + (n-1)sin^2\Phi(e^t)]e^{-2t}e^{(n-1)t}e^t dt$$

$$= \frac{\omega_n}{2}\int_{-\infty}^{0}[\Psi'^2(t) + (n-1)sin^2\Psi(t)]\, e^{(n-2)t}dt \quad (2.4)$$

The differential equation (2.3) can be written in damped pendulum equation form as follows.
Set $q(t) = 2\Psi(t) - \pi$ and $p(t) = q'(t)$, then

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ -(n-2)p - (n-1)\sin q \end{pmatrix} \quad (2.5)$$

The critical points of this system, i.e. when $p'(t) = q'(t) = 0$, imply critical points in phase plane $(\Psi(t), \Psi'(t))$, which are $(0,0)$, $(\pi, 0)$, and $\left(\frac{\pi}{2}, 0\right)$.
Here, we have to study those solutions corresponding to harmonic maps with finite energy. In this analogy the equator corresponds to the stable respoint, whereas the northpole and the southpole correspond to unstable restpoint.

Consider a function: $V(t) = (\Psi'(t))^2 - (n-1)\sin^2\Psi(t)$ (2.6)

Then

$$V'(t) = 2\Psi'(t).\Psi''(t) - 2(n-1)\,\Psi'(t).\sin\Psi(t).\cos\Psi(t)$$

$$= 2\Psi'(t).\left[\Psi''(t) - \frac{(n-1)}{2}\sin 2\,\Psi(t)\right]$$

By (2.3), $\Psi''(t) - \frac{n-1}{2}\sin 2\Psi(t) = -(n-2)\Psi'(t)$, thus

$$V'(t) = 2\Psi'(t).[-(n-2)\Psi'(t)]$$

$$= -2(n-2)(\Psi'(t))^2 \quad (2.7)$$

for $n \geq 2$, $V'(t) \leq 0$
Since Ψ: $(-\infty, 0] \to [0, \pi]$, then $\sin^2\Psi(t) \to 0$
This means $V(t) \geq 0$
These properties, i.e. $V(t) \geq 0$ and $V'(t) \leq 0$, satisfy the properties of a Lyapunov function. Therefore, $V(t)$ is a Lyapunov function for the differential equation.
Equivalently, for $V(t)e^{(n-2)t}$, $(V(t)e^{(n-2)t})'$ can be written as

$$\left(V(t)e^{(n-2)t}\right)' = -(n-2)\left[\Psi'^2(t) + (n-1)\sin^2\Psi(t)\right]e^{(n-2)t} \quad (2.8)$$

For $E(u) < \infty$,

$$\lim_{t\to-\infty}\inf\left[\Psi'^2(t) + (n-1)\sin^2\Psi(t)\right]e^{(n-2)t} = 0$$

Then because of (2.8)

$$\lim_{t\to-\infty} V(t)e^{(n-2)t} = 0 \quad (2.9)$$

and by partial integration, $E(u)$ can be written as:

$$E(u) = \frac{\omega_n}{2}\int_{-\infty}^{0}\frac{\left(V(t)e^{(n-2)t}\right)'}{-(n-2)}dt$$
$$= -\frac{\omega_n}{2(n-2)}\int_{-\infty}^{0} V'(t)e^{(n-2)t} + (n-2)V(t)e^{(n-2)t}dt$$
$$= -\frac{\omega_n}{2(n-2)}\left[\int_{-\infty}^{0} V'(t)e^{(n-2)t}dt + \int_{-\infty}^{0}(n-2)V(t)e^{(n-2)t}dt\right]$$

It is clear that

$$\int_{-\infty}^{0} V'(t)e^{(n-2)t}dt = V(t)e^{(n-2)t}\Big|_{-\infty}^{0} - \int_{-\infty}^{0}(n-2)V(t)e^{(n-2)t}dt$$

Thus,

$$E(u) = -\frac{\omega_n}{2(n-2)}\left[V(t)e^{(n-2)t}\Big|_{-\infty}^{0}\right]$$

Since $\lim_{t\to-\infty} V(t)e^{(n-2)t} = 0$

$$E(u) = -\frac{\omega_n}{2(n-2)}[V(0)]$$
$$= -\frac{\omega_n}{2(n-2)}\left[\left(\left(\Psi'(0)\right)^2 - (n-1)\sin^2\Psi(0)\right)\right]$$
$$= \frac{\omega_n(n-1)}{2(n-2)}\left[\sin^2\Psi(0) - \frac{{\Psi'}^2(0)}{n-1}\right] \quad (2.10)$$

These results can be used to classify the rotationally symmetric maps of finite energy as follows.

Lemma 1

Let $\Phi(r) = \Psi(\ln r)$ be the radius function of a rotationally symmetric harmonic map u with finite energy. Then the following cases are possible:

For *n* = 2

(i) $2\arctan(c.r) = \Phi(r)$ or $\pi - \Phi(r)$ with some constant $c \geq 0$

For $n \geq 3$, either

(ii) Ψ is constant with values 0, $\frac{\pi}{2}$ or π,

or

(iii) Ψ is extendable to a solution of the differential equation

$$\Psi''(t) + (n-2)\Psi'(t) - \frac{n-1}{2}\sin 2\Psi(t) = 0$$

and

$\lim_{t\to-\infty}\left(\Psi(t), \Psi'(t)\right) = (0,0)$ or $(\pi, 0)$

$\lim_{t\to\infty}\left(\Psi(t), \Psi'(t)\right) = \left(\frac{\pi}{2}, 0\right)$

Proof

We extend Ψ to a solution on R, which is possible due to the linear growth of the right hand side of (2.5).

#For $n = 2$

$n = 2$ in (2.9) gives $\lim_{t\to-\infty} V(t)e^0 = 0$ which implies $\lim_{t\to-\infty} V(t) = 0$,

and therefore the equation in (3.8) becomes

$$\left(\Psi'(t)\right)^2 - \sin^2\Psi(t) = 0$$
$$\Leftrightarrow \Psi'(t) = \pm\sin\Psi(t)$$
$$\Psi'(t) = +\sin\Psi(t)$$
$$\frac{d\Psi}{dt} = \sin\Psi(t)$$

By using integration by part, this can be written as

$$\int \frac{1}{\sin \Psi}\, d\Psi = \int dt$$

Since $\int \frac{1}{\sin \Psi}\, d\Psi = \ln \left|\tan \frac{\Psi}{2}\right| + k$
which means
$\ln \left|\tan \frac{\Psi}{2}\right| = t + k$, for some constant k
$\tan \frac{\Psi}{2} = e^t . c\,, \;\; c = e^k$
$*$ Thus, $\Psi(t) = 2 \arctan(e^t . c)$
$\Psi'(t) = -\sin \Psi(t) = \sin\big(\pi - \Psi(t)\big)$
$\frac{d\Psi}{dt} = \sin(\pi - \Psi)$
By using integration by part, this can be written as

$$\int \frac{1}{\sin(\pi - \Psi)}\, d\Psi = \int dt$$

Since $\int \frac{1}{\sin(\pi-\Psi)}\, d\Psi = \ln \left|\tan \frac{\pi-\Psi}{2}\right| + k$
which means
$\ln \left|\tan \frac{\pi-\Psi}{2}\right| = t + k$, for some constant k
$\tan \frac{\pi-\Psi}{2} = e^t . c\,, \;\; c = e^k$
$**$ Thus, $\pi - \Psi(t) = 2 \arctan(e^t . c)$
$\therefore$ $*$ and $**$ are equivalent to case (i).
For $n \geq 3$,
$n \geq 3$ in (2.9), which implies $\lim_{t\to-\infty} V(t) = 0$, and in (2.7), which implies $V'(t) < 0$, give: $\Psi'^2(t) < (n-1). \sin^2 \Psi(t)$
This ensures that the phase curve in $\big(\Psi(t), \Psi'(t)\big)$ is bounded.
By the definition of Lyapunov function and limit cycles, if $V'(t) < 0$, then the critical points of the differential system are stable. This means all the trajectories in phase portrait $\big(\Psi(t), \Psi'(t)\big)$ converge to the critical points. This is equivalent to case (ii).
Excluding case (ii) gives

$$\lim_{t\to-\infty} \Psi(t) = 0 \text{ or } \pi \quad \text{and} \quad \lim_{t\to\infty} \Psi(t) = \frac{\pi}{2}$$

which are equivalent to case (iii).

It is clear that $\Phi(r) = \Psi(\ln r)$ satisfying one of the cases (i) – (iii) lead to harmonic maps with finite energy. ∎

3. Stability

The result for $n = 2$ in lemma 1 explicitly shows all smooth rotationally symmetric harmonic maps with finite energy.
Whereas for $n \geq 3$, the result in lemma 1 shows that they are directly related to the trajectories of the equation connecting the critical points $(0,0)$ and $(\pi, 0)$ with $\left(\frac{\pi}{2}, 0\right)$ in the phase plane $\big(\Psi(t), \Psi'(t)\big)$.
Now consider the critical points in (q,p)-plane. They are $(-\pi, 0), (0,0)$ and $(\pi, 0)$. For $n \geq 3$, the critical points can be restricted to $(-\pi, 0)$ and $(0,0)$ corresponding to the northpole and the equator and their connecting trajectories.
The behavior of the system (2.5) in the neighborhood of the critical points is determined by the linearized system as follows.

Linearize the system around critical point

A Jacobian matrix: $\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix}$

Then, $J = \begin{pmatrix} \frac{\partial f}{\partial q} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial q} & \frac{\partial g}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(n-1)\cos q & -(n-2) \end{pmatrix}$ (3.1)

For (0, 0)

$J = \begin{pmatrix} 0 & 1 \\ -(n-1) & -(n-2) \end{pmatrix}$

with eigenvalues

$\begin{vmatrix} 0-\lambda & 1 \\ -(n-1) & -(n-2)-\lambda \end{vmatrix} = 0$

$-\lambda(-(n-2)-\lambda) + (n-1) = 0$

$\lambda^2 + (n-2)\lambda + (n-1) = 0$

$\lambda_n^{\pm} = \dfrac{-(n-2) \pm \sqrt{(n-2)^2 - 4(n-1)}}{2}$

$= \dfrac{-(n-2) \pm \sqrt{n^2 - 8n + 8}}{2}$

$$\lambda_n^+ = \frac{-(n-2)+\sqrt{n^2-8n+8}}{2} \quad \text{and} \quad \lambda_n^- = \frac{-(n-2)-\sqrt{n^2-8n+8}}{2} \qquad (3.2)$$

And for $(-\pi, 0)$

$J = \begin{pmatrix} 0 & 1 \\ (n-1) & -(n-2) \end{pmatrix}$

With eigenvalues

$\begin{vmatrix} 0-\Lambda & 1 \\ (n-1) & -(n-2)-\Lambda \end{vmatrix} = 0$

$-\Lambda(-(n-2)-\Lambda) - (n-1) = 0$

$\Lambda^2 + (n-2)\Lambda - (n-1) = 0$

$\Lambda_n^{\pm} = \dfrac{-(n-2) \pm \sqrt{(n-2)^2 + 4(n-1)}}{2}$

$= \dfrac{-(n-2) \pm \sqrt{n^2}}{2}$

$$\Lambda_n^+ = \frac{-(n-2)+n}{2} = \frac{2}{2} = 1 \text{ and } \Lambda_n^- = \frac{-(n-2)-n}{2} = \frac{-2n+2}{2} = -n+1 \qquad (3.3)$$

Recall that the behavior and stability of the critical points in phase plane satisfy following properties [4]:

(i) If the eigenvalues of the linearized matrix are ***real and distinct***, then the solution of the system will be:
- ***stable*** whenever both of the eigenvalues are both negative. The critical point is a *stable improper node*.
- ***unstable*** whenever one or both of the eigenvalues are positive. The critical point is a *saddle*.

(ii) If the eigenvalues of the linearized matrix are ***complex***, then the solution for the system will be:
- ***asymptotically stable*** if the real part is negative. The critical point is a *stable spiral or focus*.
- ***stable*** if the real part equals to zero. The critical point is a *center*.
- ***unstable*** if the real part is positive. The critical point is a *unstable spiral or focus*.

(iii) If the eigenvalues of the linearized matrix are ***real and equal***, then the solution for the system will be:
- ***asymptotically stable*** if the eigenvalue is negative.
- ***unstable*** the eigenvalue greater than and equals to zero.

The critical point is a *proper node* if the eigenvalues have two linearly independent eigenvectors, and a *degenerate or inflected node* if the eigenvalues have one corresponding eigenvector.

Therefore, the stability and behavior of the critical points in this case can be determined as follows:

For $(0,0)$ with eigenvalues: $\lambda_n^{\pm} = \frac{-(n-2)\pm\sqrt{n^2-8n+8}}{2}$,

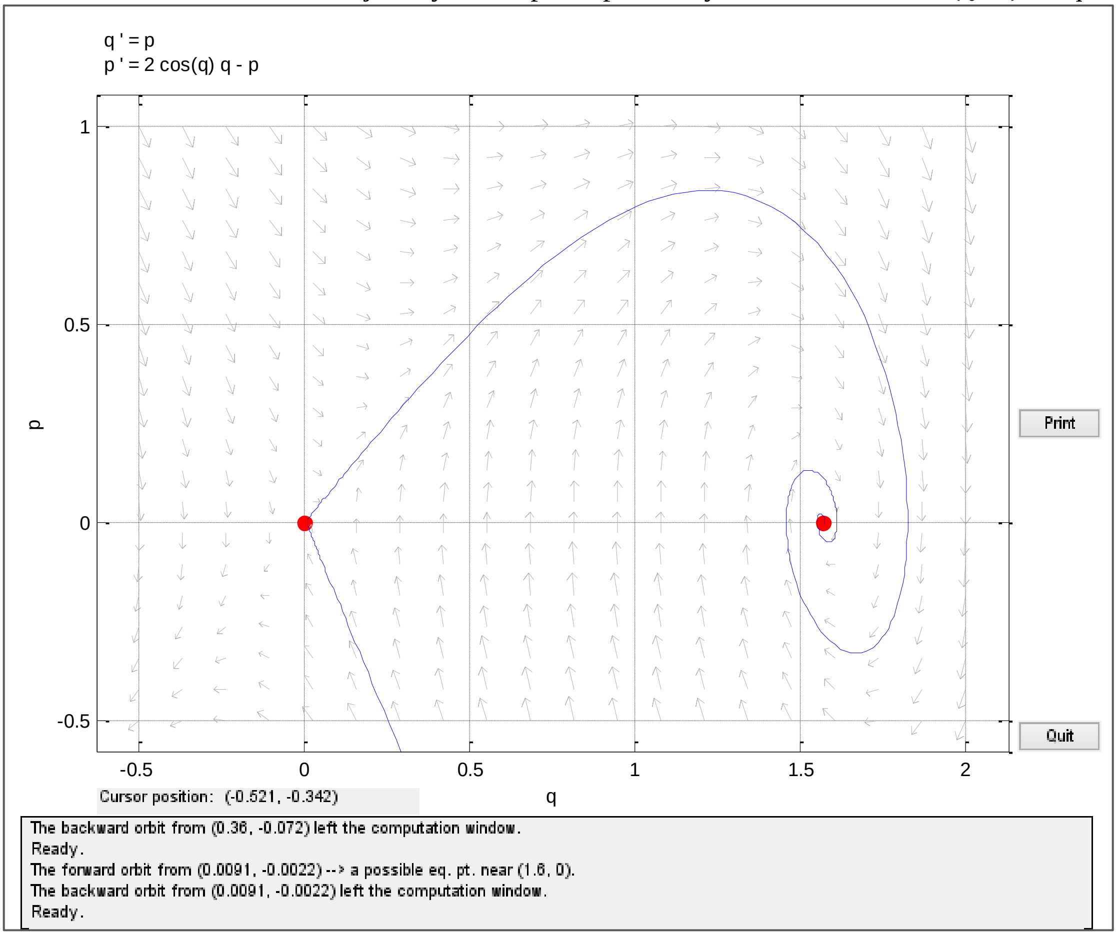
- if $n = 2$, $\lambda_n^{\pm} = \pm 2i$, which are purely imaginary, then $(0,0)$ is a *center*, which means the trajectories in phase plane go around $(0,0)$.
- if $3 \leq n \leq 6$, $\lambda_n^{\pm}$ are complex with negative real part, then $(0,0)$ is a *stable spiral or focus*, which means the trajectories are spiraling into $(0,0)$.
- if $n \geq 7$, $\lambda_n^{\pm}$ are distinct real and negative, then $(0,0)$ is an *stable improper node*, which means the trajectories are sinking into $(0,0)$. The directions of the trajectories depend on the eigenvector, i.e. they sink into $(0,0)$ from the directions of eigenvector that corresponds to the smallest eigenvector.

For $(-\pi, 0)$ with eigenvalues: $\Lambda_n^{+} = 1$ and $\Lambda_n^{-} = -n + 1$, since $n \geq 2$, Λ_n^{-} will always be negative, then $(-\pi, 0)$ will always be a *saddle.*

It is clear that there exists an invariant curve in the trajectories between the critical points. For example for $n = 3$, the qp-plane can be seen in picture 1. As it is shown in the picture, there exists exactly one invariant curve from $(-\pi, 0)$ to $(0,0)$. The other curves of trajectories can be assumed as a translation in the parameter t. Let $(Q, P): R \to R^2$ be such a trajectory, then

$$\lim_{t\to-\infty} e^{-t}.P(t) = c \tag{3.4}$$

exists and the curve of each trajectory lines up independently, thus c determines (Q, P) uniquely.



Picture 1. Trajectories in *qp*-plane

4. Conclusion

Study of rotationally symmetric maps with finite energy by considering map $u: B^n \to S^n$ that can be written in the form

$$u(x) = (g(x).\sin\varphi(x), \cos\varphi(x))$$

can be reduced into a damped pendulum equation system as follows.

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ -(n-2)p - (n-1)\sin q \end{pmatrix}$$

The solution of this system for $n = 2$ explicitly shows all smooth rotationally symmetric harmonic maps with finite energy. Whereas for $n \geq 3$, the result shows that they are directly related to the trajectories of the equation connecting the critical points $(0,0)$ and $(\pi, 0)$ with $\left(\frac{\pi}{2}, 0\right)$ in the phase plane $\big(\Psi(t), \Psi'(t)\big)$.

Consider the critical points in (q, p)-plane: $(-\pi, 0), (0,0)$ and $(\pi, 0)$. For $n \geq 3$, the critical points can be restricted to $(-\pi, 0)$ and $(0,0)$ corresponding to the northpole and the equator and their connecting trajectories.

Therefore, the stability and behavior of the critical points in this case can be determined as follows:

For $(0,0)$,

- if $n = 2$, $(0,0)$ is a *center*, which means the trajectories in phase plane go around $(0,0)$.
- if $3 \leq n \leq 6$, $(0,0)$ is a *stable spiral or focus*, which means the trajectories are spiraling into $(0,0)$.
- if $n \geq 7$, $(0,0)$ is an *stable improper node*, which means the trajectories are sinking into $(0,0)$. The directions of the trajectories depend on the eigenvector, i.e. they sink into $(0,0)$ from the directions of eigenvector that corresponds to the smallest eigenvector.

For $(-\pi, 0)$, it will always be a *saddle*.

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