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Book Embedding of Infinite Family $\binom{2h+3}{2}$-Crossing-Critical Graphs for $h = 1$ with Rational Average Degree $r \in (3.5, 4)$

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A B S T R A C T

A principal tool used in construction of crossing-critical graphs are tiles. In the tile concept, tiles can be arranged by gluing one tile to another in a linear or circular fashion. The series of tiles with circular fashion form an infinite graph family. In this way, the intersection number of this family of graphs can be determined. In this research, has been formed an infinite family graphs $Q_{(1,s,b)}(n)$ with average degree r between 3.5 and 4. The graph formed by gluing together many copies of the tile $P_{(1,s,b)}$ in circular fashion, where the tile $P_{(1,s,b)}$ consist of two identical pieces of tile. And then, the graph embedded into the book to determine the *pagenumber* that can be formed. When embed graph into book, the vertices are put on a line called the spine and the edges are put on half-planes called the pages. The results obtained show that the graph $Q_{(1,s,b)}(n)$ has 10-crossing-critical and book embedding of graph has 4-page book.

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A B S T R A K

Alat utama yang digunakan dalam mengkonstruksi graf perpotongan kritis adalah ubin. Dalam konsep ubin, ubin dapat disusun dengan merekatkan satu ubin ke ubin lainnya secara linier atau melingkar. Rangkaian ubin yang disusun secara melingkar membentuk keluarga graf tak hingga. Dengan cara ini, angka perpotongan dari keluarga graf ini dapat ditentukan. Dalam penelitian ini telah dibentuk graf keluarga tak hingga dengan derajat rata-rata antara 3,5 dan 4. Graf dibentuk dengan merekatkan banyak salinan ubin secara melingkar, dimana ubin ini terdiri dari dua buah ubin yang identik. Kemudian graf tersebut disematkan ke dalam buku untuk ditentukan jumlah halaman yang dapat dibentuk. Saat menyematkan graf ke dalam buku, verteks diletakkan pada garis yang disebut tulang belakang buku dan edges diletakkan pada bidang setengah yang disebut halaman. Hasil yang diperoleh menunjukkan bahwa graf memiliki 10-perpotongan kritis dan penyematan buku dari graf memiliki 4 halaman buku.

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1. INTRODUCTION

Construction of mathematical models can be made in various ways using different mathematical problems. One mathematical model that is well known and can cover a variety of problems is graph theory [15]. Graph theory is a part of discrete mathematics whose application is used to present many practical problems which are highly related to everyday life, such as modeling the types of relationships and process dynamics in the fields of physics, biology, chemistry, and social systems.

Graphs are often used to present an object and its relationship with other objects. The first graph theory was discovered by Leonhard Euler, a Swiss Mathematician, when solving the Königsberg bridge problem. From these problems, graph theory developed widely. At present, there are so many graph models that are used to find solutions in solving a problem.

In a drawing of a graph G, the vertices of G are points and the edges are simple curves joining their endvertices. The most common way to represent a graph is a diagram. An obvious fact about these diagrams is that sometimes the edges cross at the points that do not belong to the graph. In other words, in diagramatic incarnations, the edges may occasionally meet at the points that are not the nodes of the graph. For planar graphs, there is always a representation that avoids such crossings at all. But not all graphs are planar. And for the latter kind, an important characteristics is their crossing number.

The crossing number of a graph G is the minimum number of crossing points of edges in a drawing of G in the plane. The crossing number of a graph is often denoted as $cr(G)$. Reducing the number of crossings in a drawing of graph is considered one of the most important drawing aesthetics. The reasons for sparse graph to have many crossing in any drawing are structural. Structure differences can affect the number of intersections of a graph. These reasons can be understood via so called k-crossing-critical graphs. For $k \in \mathbb{N}$, a graph G is called k-crossing-critical, if $cr(G) \geq k$, but for every edge e of G, $cr(G - e) < k$ [1].

According to [1], a principal tool used in construction of crossing-critical graphs are tiles. Tiles are essentially graphs equipped with two sequences of vertices that are identified among tiles or within a tile in order to, respectively, form new tiles or tiled graphs.

The tile concept was first introduced by [9] in research about "Crossing Numbers of Sequences of Graphs in case Planar Tiles". A tile is a 3 tuple $T = (G, L, R)$ which is connected graph together with two specified sequences of vertices, the left walls (L) and right walls (R). A tile drawing of a tile T is a drawing of the unit square with the left wall in order down the left hand side and the right wall in order down the right hand side. Tiles can be arranged by gluing one tile to another and arranged in a linear or circular fashion. The tile T^n is obtained by gluing n copies of T in linear fashion, while the graph $\circ(T^n)$ is obtained by gluing n copies of T in a circular fashion. The series of tiles with circular fashion form an infinite graph family. In this way, the intersection number of this family of graphs can be determined.

A book embedding of a graph is a linear ordering of its vertices, and a partition of its edges into non-crossing sets (called pages) [6]. Embedding graphs in books is a fundamental issue in graph theory that has received considerable attention. In a book embedding, the vertices of a graph are restricted to a line, referred to as the spine of the book, and the edges are drawn at different half-planes delimited by the spine, called pages of the book. The task is to find a so-called linear order of the vertices along the spine and an assignment of the edges of the graph to the pages of the book, so that no two edges of the same page cross. The *book thickness* or *pagenumber* of a graph is the smallest number of pages that are required by any book embedding of the graph [2].

In [8] research, they describe a method of creating an infinite family of crossing-critical graphs from a single small planar map, the tile, by gluing together many copies of the tile together in a circular fashion. They gave as example the infinite family of $\binom{2h+3}{2}$-crossing-critical simple graphs, with positive integer h and having rational average degree $r \in (3.5, 4)$.

In previous research by [11], they form the graphs which is the infinite families of $\binom{2h+3}{2}$-crossing-critical graphs with average degree between 3.5 and 4. In that research, they set $h = 0$ and get 3-crossing-critical graphs. And then, they embed the graph into book and show that *pagenumber* of book embedding $\rho n = 3$.

In this research, continuing previous research, also will be form the graphs of infinite families of $\binom{2h+3}{2}$-crossing-critical graphs with average degree r between 3.5 and 4 for $h = 1$, and the graph will be embedded into book. And then, will be seen, whether the number of pages in the book embedding is the same as the number of crossings on the graphs or not.

2. LITERATURE REVIEW

2.1. Graph Theory

Graph is defined as a pair of sets (V, E), in which case V is a non-empty set of vertices (nodes) and E is a set of edges that connects a pair of vertices [7]. According to historical records, graph theory was born in 1736 through the writings of Leonard Euler which contained an effort to solve the problem of the Königsberg bridge which is very famous in Europe, that is, through each of the seven bridges exactly once and returning to its starting place.

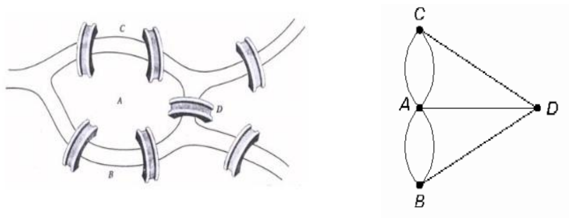

Figure 1. (a) The Königsberg Bridge; (b) Graph that presents The Königsberg Bridge

Euler was the first person who managed to find the answer to that problem by modeling this problem into graphs. Land (points connected by bridges) are expressed as points (dots) called vertices and bridges are expressed as lines called edges as shown in Figure 1.(b) [5].

A linear graph (or simple graph) $G = (V, E)$ consists of a group of objects $V = \{v_1, v_2, ...\}$ called a set of points, and another set of $E = \{e_1, e_2, ...\}$ which is a set of sides such that each side of e_k is associated with an unordered pair (v_i, v_j). Points v_i, v_j related to e_k are called the end points of the e_k side. The most common way to represent a graph is in the form of a diagram. In the diagram, the points are expressed as dots and each side is expressed as a line segment connecting each two points [14].

In a graph, as in Figure 2.(b), it is possible to have a side associated with a point pair (v_1, v_2). A side with two equal ends is called a loop. From the Figure 2.(b) should also be noted that in a graph it is possible to have more than one side associated with a pair of points. For example, e_4 and e_5 in the graph in Figure 2.(b) are associated with point pairs (v_1, v_3). Such side pairs are called parallel sides or multiple edges [14]. A graph that has no loops and parallel sides is called a simple graph. A graph that has multiple edges is called a multigraph, while a graph containing loop is called pseudo graph [4].

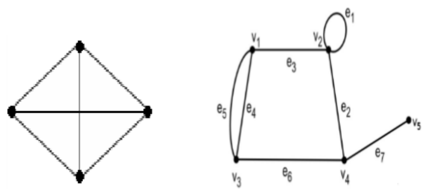

Figure 2. (a) Simple Graph; (b) Multigraf (Nonsimple Graph)

2.2. Crossing Number

The crossing number $cr(G)$ of a graph G is the minimum number of crossing points of edges in a drawing of G in the plane. For $k \in \mathbb{N}$, a graph G is called

k-crossing-critical, if $cr(G) \geq k$ but, for every edge e of G, $cr(G - e) < k$ [1].

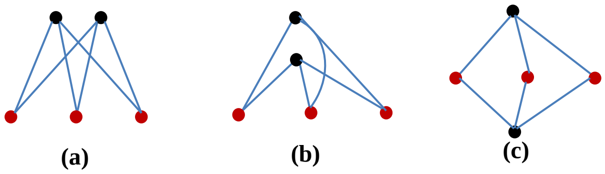

Figure 3. Crossing Number of $K_{2,3}$
(a) $Cr_{2,3} = 3$, (b) $Cr_{2,3} = 1$, (c) $Cr_{2,3} = 0$

In general, for a graph G, the minimum number of pairwise crossings of edges among all drawings of G in the plane is the crossing number of G and is denoted by $cr(G)$. Thus, $Cr_{2,3} = 3$. We remark that $Cr_{2,3} \leq 3$. It is an interesting exercise for the reader to prove that $Cr_{2,3} = 3$. At present, there is no known efficient algorithm to calculate the crossing number of an arbitrary graph. In fact, the problem of calculating the crossing number of a graph is NP-complete, so it is unlikely that such an efficient algorithm exists. Yet one might hope that the crossing number of a graph with special structure can be calculated [12].

According to [10], a graph can be drawn in many ways on the plane. A good drawing has the following properties :

- No edge crosses itself
- No pair of adjacent edges cross
- Two edges cross at most once
- No more than two edges cross at the point

2.3. Crossing-Critical Graph

A graph G with crossing number k or more is k-crossing-critical if every proper subgraph has crossing number less than k, and G is homomorphically minimal. Note that this last condition, together with Kuratowski's Theorem implies that the only 1-crossing-critical graphs are $K_{3,3}$and K_5. Remark that there are graphs G that are k-crossing-critical for some $k < cr(G)$ [13].

If G is k-crossing-critical, then G has bounded tree–width, thereby directly relating crossing–number problems to graph minors. Crossing numbers are not monoton with respect to contraction, since contracting an edge might decrease, increase, or not change the crossing number, so this is a significant relationship [13].

2.4. Average Degree

The average degree of a graph G is another measure of how many edges are in set E compared to number of vertices in set V. Because each edge is incident to two vertices and counts in the degree of both vertices. In graph theory, the degree (or valency) of a vertex of a graph is the number of edges incident to the vertex with loops counted twice. The average degree of an undirected graph are formulated by

$$2 \times \frac{|E|}{|V|}$$

2.5. Tile

A tile is a 3-tuple $T = (G, L, R)$, where

- G is connected graph;
- L is a finite sequence of vertices of $V(G)$, called the left-wall;
- R is a finite sequence of vertices of $V(G)$, called the right-wall; and
- all the vertices in L and R are distinct.

Figure 4. shows some three examples of tiles, the tiles K, M, and Z. The left-wall of each of these tiles is the sequence abc and the right-wall is the sequence def of vertices [8].

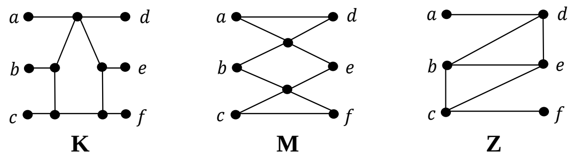

Figure 4. Tile K, M, and Z

A tile drawing of a tile $T = (G, L, R)$ is a drawing of T in the unit square $S = \{(x, y): -1 \leq x \leq 1 \; and -1 \leq y \leq 1\}$ such that: the intersection with the boundary $\{(x, y) \in S$: either $x \in \{-1,1\}$ or $y \in \{-1,1\}\}$ is precisely $L \cup R$; the vertices in L occur in the line $x = -1$, with the y-coordinates of $L[1], L[2], \dots, L[|L|]$ decreasing; and the vertices in R occur in the line $x = 1$, with the y-coordinates of $R[1], R[2], \dots, R[|R|]$ also decreasing. The tile crossing number $tcr(T)$ of a tile T is the smallest number of crossing in any tile drawing of T [9].

If $tcr(T) = 0$, a tile $T = (G, L, R)$ is *planar*, and otherwise the tile is non-planar. The twist $T' = (G, L, R')$ of $T = (G, L, R)$ is the tile obtained by reversing the order of R. A tile $T_1 = (G_1, L_1, R_1)$ is compatible with a tile $T_2 = (G_2, L_2, R_2)$ if the function $f: R_1 \to L_2$ defined by $f(R_1[i] = L_2[i]$ is bijection and $\forall v_1, v_2 \in R_1, \; v_1v_2 \in E(G) \Leftrightarrow f(v_1)f(v_2) \in E(G_2)$. A tile is self-compatible if it is compatible to itself [9].

2.6. Book Embedding

The concept of book embeddings were first introduced by Oll-mann in 1973 and Kainen. A book consist of a line called spine and some half-planes called pages, sharing the spine as a common boundary. Book embedding of a graph $G = (V, E)$ consists of a (linear) layout L of its notes along the spine l of a book (i.e., a one-to-one function from V to $\{1, \dots, n\}$) and the assignment of each edge on the pages so that two edges embedded on the same page do not intersect. The book embedding plays an important role in VLSI design, matrix computation, parallel processing, and permutation sorting. A central goal in the study of book embeddings is to find the minimum number of pages in which a graph can be embedded, this is called the *pagenumber* or *book thickness* of the graph [3].

The pagenumber of a graph G, denoted $pn(G)$, is the minimum number of pages in which it can be embedded without crossings. The pagenumber of a graph can also referred to as its book thickness, stack number or fixed outer-thickness [6].

3. RESEARCH METHODOLOGY

3.1. Time and Research Place

This research has been held in Mathematics Departments of Sam Ratulangi University from November 2019 until March 2020.

3.2. Research Methods

The method that used in this research is a literature study by searching theory references related to crossing number of graph, tile graph, and book embedding.

3.3. Research Stage

The steps to be used in completing this research are as follows:

1. Study the literature from books and journals relating to the problem.
2. Make construction of infinite family of $\binom{2h+3}{2}$-crossing-critical graphs for $h = 1$ with rational average degree $r \in (3.5, 4)$ using the tile concept.
3. Make the structure book embedding of infinite family of $\binom{2h+3}{2}$-crossing-critical graphs for $h = 1$.
4. Determine *pagenumber* $\rho n(G)$ in the book embedding of $\binom{2h+3}{2}$-crossing-critical graphs for $h = 1$.

4. RESULTS AND DISCUSSION

4.1. Construction of Tile

In this research, the tile concept is used to build an infinite family of graphs. Therefore, the first thing to do is construct tiles.

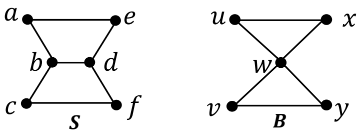

Figure 5. Tile S and Tile B

Figure 5 shows the tile $S = ((\{a,b,c,d,e,f\}, \{ab, ae, bc, bd, cf, de, df\}), (a,c), (e,f))$ and tile $B = ((\{u,v,w,x,y\}, \{uw, ux, vw, vy, wx, wy\}), (u,v), (x,y))$. A tile $S = (G_1, L_1, R_1)$ is compatible with tile $B = (G_2, L_2, R_2)$. If tile S and B is compatible, then the catenation of the both is $SB = (G_1 \cup G_2, L_1, R_2)$ where R_1 corresponds with L_2. From the Figure 6, in the catenation SB, vertices e and f of S are identified with respectively the vertices u and v of tile B. A tile is *self-compatible* if it is compatible with itself. If S be a self-compatible tile, then S^n is the catenation of n copies of S in linear fashion and $\circ(S^n)$ is the catenation of n copies of S in circular fashion where the first and the last order of tile S also put in catenation. In Figure 6, shows the tile S is self-compatible, where the vertices of R_1 (e_1 and f_1) are identified with respectively the vertices L_2 (a_1 and c_1) of the catenation SS, as well as with B.

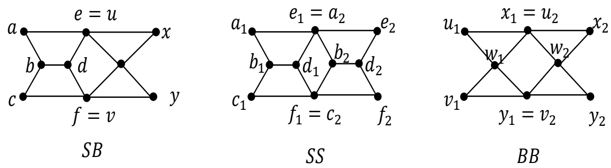

Figure 6. Catenations SB, SS, and BB

For $h = 1$, the formed tile will be added with a copy of a "long" version of S placed sequentially under the tile. Figure 9 shows the tile S_h with $h = 1$, for $S_1 = ((\{a,b,c,d,e,f,q_1,q_2,q_3,q_4\}, \{ab, ae, bc, bd, cf, de, df, cq_1, fq_3, q_1q_2, q_1q_3, q_2q_4, q_3q_4\}), (a,c,q_2), (e,f,q_4))$ and tile B_h with $h = 1$, for $B_1 = ((\{u,v,w,x,y,q_1,q_2,q_3, q_4\}, \{uw, ux, vw, vy, wx, wy, vq_1, yq_3, q_1q_2, q_1q_3, q_2q_4, q_3q_4\}), (u,v,q_2), (x,y,q_4))$, the twist $S'_1 = ((\{a,b, c,d,e,f,q_1,q_2,q_3,q_4\}, \{ab, ae, bc, bd, cf, de, df, cq_1, fq_3, q_1q_2, q_1q_3, q_2q_4, q_3q_4\}), (a,c,q_2), (q_4,f,e))$, and the twist $B'_1 = ((\{u,v,w,x,y,q_1,q_2,q_3,q_4\}, \{uw, ux, vw, vy, wx, wy, vq_1, yq_3, q_1q_2, q_1q_3, q_2q_4, q_3q_4\}), (u,v,q_2), (q_4,y,x))$. The tiles S_1 and B_1 are planar, while their twists S'_1 and B'_1 are not planar. For the twist S'_1 has tile crossing number $tcr(S'_1) = 10$ and $tcr(B'_1) = 12$. According to the rules of crossing number, to get the smallest crossing number, the twist can only be through the tile S.

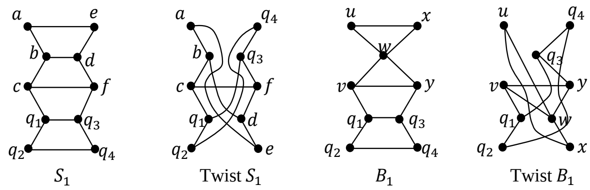

Figure 7. Tiles S_1 and B_1 with Their Twist S'_1 and B'_1

4.2. Infinite Family of $\binom{2h+3}{2}$-Crossing-Critical Graphs for $h = 1$ with Rational Average Degree $r \in (3.5, 4)$

In this research, to built an infinite family of graphs, refer the theory of tile that has been introduced by [8]. They introduced the theory of tile which is two small pieces of graph with certain conditions and can be glued together in a circular fashion to built an infinite family of graphs. They define the infinite family of $\binom{2h+3}{2}$-crossing-critical graphs for h is positive integer with having rational average degree between 3.5 and 4.

In this research, will be set $h = 1$ and define $s \geq 1$, $b \geq 0$, $s + b \geq 2$, for the tile $P_{(1,s,b)}$ is a tile consisting of tile S and B which are tiles in Figure 11, and added with $h = 1$ in linear fashion, and the graph $Q_{(1,s,b)}(n) = \circ(P'P^{n-1})$, where P' is twist circular catenation of $P_{(1,s,b)}$ and P^{n-1} is the catenation of $n-1$ copies of $P_{(1,s,b)}$.

4.2.1 Tile $P_{(1,s,b)}$

For the example will define $s = 3$ and $b = 3$, thus forming a tile $P_{(1,3,3)}$ in linear fashion (Figure 8). The tile P are planar, where without crossing. To get crossing, the order of R in tile P must be reversed. If the twist is through the tile S, the tile has 10-crossing-critical (Figure 9). While, if the twist is through the tile B, the tile has 12-

crossing-critical (Figure 10). Therefore, according to the rules of tile crossing number, the tile smallest number of crossing is used, which is the twist through the tile S.

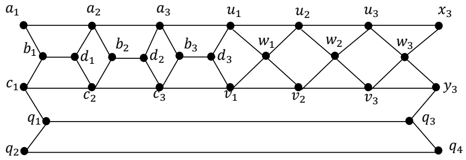

Figure 8. Tile $P_{(1,3,3)}$

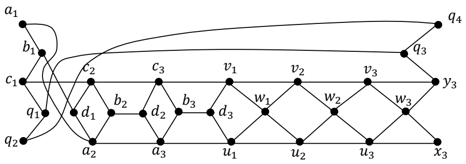

Figure 9. Tile $P'_{(1,3,3)}$ with Twist Through S

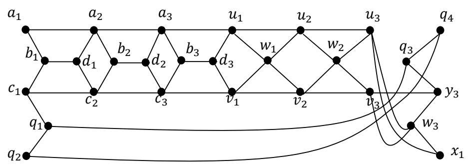

Figure 10. Tile $P'_{(1,3,3)}$ with Twist Through B

4.2.2 The Infinite Family Graph $Q_{(1,s,b)}(n)$

For $s \geq 1$, $b \geq 0$, and $s + b \geq 2$, the graph $Q_{(1,s,b)}(n) = \circ(P'P^{n-1})$ has vertices $a_{1m}, b_{1m}, c_{1m}, d_{1m}, \ldots, a_{sm}, b_{sm}, c_{sm}, d_{sm}, \ldots, u_{1m}, v_{1m}, w_{1m}, \ldots, u_{bm}, v_{bm}, w_{bm}, q_{11}, q_{21}, q_{31}, \ldots, q_{1m}, q_{2m}, q_{3m}$ for $(1 \leq m \leq n)$, and the edges $a_{km}b_{km}$, $b_{km}c_{km}$, $b_{km}d_{km}(1 \leq k \leq s)(1 \leq m \leq n)$, $a_{km}a_{(k+1)m}, c_{km}c_{(k+1)m}, d_{km}a_{(k+1)m}, d_{km}c_{(k+1)m}$ $(1 \leq k \leq s-1)$ $(1 \leq m \leq n)$, $u_{km}w_{km}$, $v_{km}w_{km}$ $(1 \leq k \leq b)$ $(1 \leq m \leq n)$, $u_{km}u_{(k+1)m}$, $v_{km}v_{(k+1)m}$, $w_{km}v_{(k+1)m}$, $w_{km}u_{(k+1)m}$ $(1 \leq k \leq b-1)(1 \leq m \leq n)$, $a_{sm}u_{1m}$, $d_{sm}u_{1m}$, $c_{sm}v_{1m}$, $d_{sm}v_{1m}$, $q_{1m}q_{3m}$, $q_{1m}q_{2m}$, $c_{1m}q_{1m}$ $(1 \leq m \leq n)$, $w_{bm}c_{1(m+1)}$, $v_{bm}c_{1(m+1)}$, $c_{1(m+1)}q_{3m}$ $(1 \leq m \leq n-1)$, $u_{b(m+1)}a_{1(m+2)}$, $w_{b(m+1)}a_{1(m+2)}$, $q_{3(m+1)}q_{2(m+2)}$, $q_{2(m+1)}q_{2(m+2)}$ $(1 \leq m \leq n-2)$, $q_{2n}q_{21}$, $q_{3n}q_{21}$, $q_{3n}c_{11}$, $u_{bn}a_{11}$, $w_{bn}a_{11}$, $w_{bn}c_{11}$, $v_{bn}c_{11}$, $q_{21}a_{12}$, $q_{31}a_{12}$, $u_{21}q_{22}$, $w_{21}q_{22}$.

The average degree of $Q_{(1,s,b)}(n)$ is

$$r = \frac{14s + 12b + 12}{4s + 3b + 3}$$

Figure 11 shows $Q_{(1,3,3)}(2)$ as the example of infinite family 10-crossing-critical graphs with rational average degree $r \in (3.5,4)$ where is the catenation of $P'_{(1,s,b)}$ (twist of tile $P_{(1,s,b)}$) and tile $P_{(1,s,b)}$. The average degree of $Q_{(1,3,3)}(2)$ and $Q_{(1,3,3)}(3)$ is 3.75.

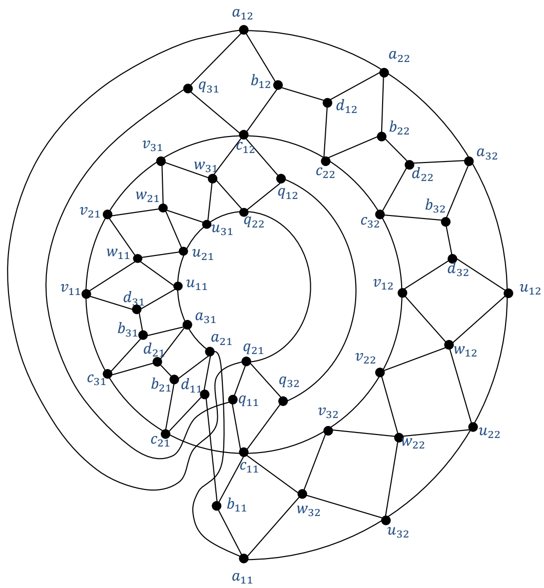

Figure 11. Graph $Q_{(1,3,3)}(2)$

4.3. Book Embedding of $Q_{(1,s,b)}(n)$

In this section, the graph $Q_{(1,s,b)}(n)$ will be embedded into the book. When embedded into a book, vertices in a graph are arranged horizontally in a straight line (called the spine of the book) and edges are defined as book pages that connect each vertex by drawing it half-circle.

The result obtained, the smallest number of pages book that can be formed of $Q_{(1,s,b)}(n)$ is 4-page book, with the arrangement of the vertices is $a_{11}, b_{11}, c_{11}, d_{11}, a_{21}, b_{21}, c_{21}, d_{21}, \ldots, a_{s1}, b_{s1}, c_{s1}, d_{s1}, u_{11}, v_{11}, w_{11}, u_{21}, v_{21}, w_{21}, \ldots, u_{b1}, v_{b1}, w_{b1}, \ldots, a_{1n}, b_{1n}, c_{1n}, d_{1n}, a_{2n}, b_{2n}, c_{2n}, d_{2n}, \ldots, a_{sn}, b_{sn}, c_{sn}, d_{sn}, u_{1n}, v_{1n}, w_{1n}, u_{2n}, v_{2n}, w_{2n}, \ldots, u_{bn}, v_{bn}, w_{bn}, \ldots, q_{3n}, q_{2n}, q_{1n}, q_{3(n-1)}, q_{2(n-1)}, q_{1(n-1)}, \ldots, q_{32}, q_{22}, q_{12}, q_{31}, q_{21}, q_{11}$. On page 1, put the following edges: $a_{km}b_{km}$, $b_{km}c_{km}$, $b_{km}d_{km}$ $(1 \leq k \leq s)$ $(1 \leq m \leq n)$, $a_{km}a_{(k+1)m}$, $d_{km}a_{(k+1)m}$ $(1 \leq k \leq s-1)$ $(1 \leq m \leq n)$, $u_{km}w_{km}$, $v_{km}w_{km}$ $(1 \leq k \leq b)(1 \leq m \leq n)$, $u_{km}u_{(k+1)m}$, $w_{km}u_{(k+1)m}$ $(1 \leq k \leq b-1)(1 \leq m \leq n)$, $a_{sm}u_{1m}$, $d_{sm}u_{1m}$, $q_{1m}q_{2m}$, $q_{1m}q_{3m}$ $(1 \leq m \leq n)$, $u_{b(m+1)}a_{1(m+2)}$, $w_{b(m+1)}a_{1(m+2)}$ $(1 \leq m \leq n-2)$, $u_{bn}a_{11}$, $w_{bn}a_{11}$. On page 2, put the following edges: $c_{km}c_{(k+1)m}$, $d_{km}c_{(k+1)m}$ $(1 \leq k \leq s-1)(1 \leq m \leq n)$, $v_{km}v_{(k+1)m}$, $w_{km}v_{(k+1)m}$ $(1 \leq k \leq b-1)(1 \leq m \leq n)$, $c_{sm}v_{1m}$, $d_{sm}v_{1m}$ $(1 \leq m \leq n)$, $w_{bm}c_{1(m+1)}$, $v_{bm}c_{1(m+1)}$ $(1 \leq m \leq n-1)$, $q_{3(m+1)}q_{2(m+2)}$, $q_{2(m+1)}q_{2(m+2)}$ $(1 \leq m \leq n-2)$, $q_{2n}q_{21}$, $q_{3n}q_{21}$, $q_{3n}c_{11}$, $w_{bn}c_{11}$, $v_{bn}c_{11}$. On page 3, put the following edges: $c_{1m}q_{1m}$ $(1 \leq m \leq n)$, $c_{1(m+1)}q_{3m}$ $(1 \leq m \leq n-1)$, $q_{21}a_{12}$, $q_{31}a_{12}$. On page 4, put the following edges: $u_{31}q_{22}$, $w_{31}q_{22}$. Therefore, it can be concluded that $\rho n(Q_{(1,s,b)}) = 4$.

Figure 12 shows a book embedding of graph $Q_{(1,3,3)}(2)$, where of each other have 4-page book. On figure, page 1 are expressed as black line, page 2 are expressed as green line, page 3 are expressed as red line, and page 4 are expressed as blue line.

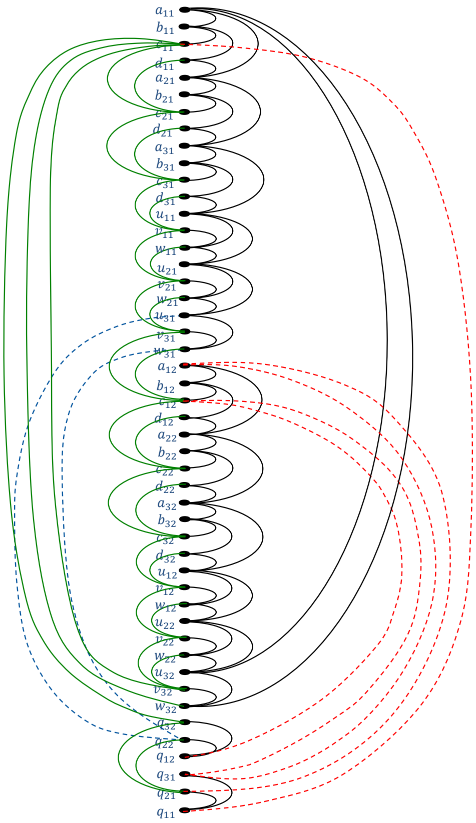

Figure 12. Book Embedding of $Q_{(1,3,3)}(2)$

5. CONCLUSION AND SUGGESTION

5.1. Conclusion

Based on the result of the study, it can be concluded that:

The infinite family of $\binom{2h+3}{2}$-crossing-critical graphs for $h = 1$ has 10-crossing-critical. And, the shape of infinite family graphs $Q_{(h,s,b)}(n)$ for $h = 1$ where $Q_{(h,s,b)}(n) = \circ\,(P'P^{n-1})$ has vertices a_{1m}, b_{1m}, c_{1m}, d_{1m}, ..., a_{sm}, b_{sm}, c_{sm}, d_{sm}, ..., u_{1m}, v_{1m}, w_{1m}, ..., u_{bm}, v_{bm}, w_{bm}, q_{11}, q_{21}, q_{31}, ..., q_{1m}, q_{2m}, q_{3m} for $(1 \le m \le n)$, and the edges $a_{km}b_{km}$, $b_{km}c_{km}$, $b_{km}d_{km}$ $(1 \le k \le s)(1 \le m \le n)$, $a_{km}a_{(k+1)m}$, $c_{km}c_{(k+1)m}$, $d_{km}a_{(k+1)m}$, $d_{km}c_{(k+1)m}$ $(1 \le k \le s-1)(1 \le m \le n)$, $u_{km}w_{km}$, $v_{km}w_{km}$ $(1 \le k \le b)(1 \le m \le n)$, $u_{km}u_{(k+1)m}$, $v_{km}v_{(k+1)m}$, $w_{km}v_{(k+1)m}$, $w_{km}u_{(k+1)m}$ $(1 \le k \le b-1)(1 \le m \le n)$, $a_{sm}u_{1m}$, $d_{sm}u_{1m}$, $c_{sm}v_{1m}$, $d_{sm}v_{1m}$, $q_{1m}q_{3m}$, $q_{1m}q_{2m}$, $c_{1m}q_{1m}$ $(1 \le m \le n)$, $w_{bm}c_{1(m+1)}$, $v_{bm}c_{1(m+1)}$, $c_{1(m+1)}q_{3m}$ $(1 \le m \le n-1)$, $u_{b(m+1)}a_{1(m+2)}$, $w_{b(m+1)}a_{1(m+2)}$, $q_{3(m+1)}q_{2(m+2)}$, $q_{2(m+1)}q_{2(m+2)}$ $(1 \le m \le n-2)$, $q_{2n}q_{21}$, $q_{3n}q_{21}$, $q_{3n}c_{11}$, $u_{bn}a_{11}$, $w_{bn}a_{11}$, $w_{bn}c_{11}$, $v_{bn}c_{11}$, $q_{21}a_{12}$, $q_{31}a_{12}$, $u_{21}q_{22}$, $w_{21}q_{22}$.

The structure the book embedding of infinite family graphs $Q_{(h,s,b)}(n)$ for $h = 1$ forms 4-page book. Therefore, *pagenumber* of book embedding $Q_{(1,s,b)}(n)$ is $\rho n(G) = 4$.

5.2. Suggestion

Hopefully, through this research, the researchers can further develop theories about graphs especially tile concept and find the structure book embedding of infinite family graph with $h > 1$ or with other concept.

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