Rectilinear Monotone r-Regular Planar Graphs for $r = \{3, 4, 5\}$

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Abstract

A graph *G* consists of non-empty set of *vertex/vertices* (also called *node/nodes*) and the set of lines connecting two vertices called *edge/edges*. The vertex set of a graph *G* is denoted by *V(G)* and the edge set is denoted by *E(G)*. A *Rectilinear Monotone r-Regular Planar Graph* is a simple connected graph that consists of vertices with same degree and horizontal or diagonal straight edges without vertical edges and edges crossing. This research shows that there are infinite family of rectilinear monotone r-regular planar graphs for r = 3and r = 4. For r = 5, there are two drawings of rectilinear monotone r-regular planar graphs with 12 vertices and 16 vertices.

Keywords: Monotone Drawings, Planar Graphs, Rectilinear Graphs, Regular Graphs.

Graf Rectilinier Monoton r-Teratur Planar untuk $r = \{3, 4, 5\}$

Abstrak

Suatu graf G terdiri dari himpunan tak kosong simpul dan himpunan garis yang menghubungkan dua simpul yang disebut sisi. Himpunan simpul suatu graf G diberi simbol V(G) dan himpunan sisinya diberi simbol E(G). Graf rectilinier monoton r-teratur planar adalah suatu graf terhubung sederhana yang terdiri atas simpul-simpul berderajat sama dan sisi-sisi yang berupa garis horizontal atau diagonal tanpa garis vertikal dan perpotongan sisi. Terdapat famili tak hingga dari graf rectilinier monoton r-teratur planar untuk r = 3 dan r = 4. Untuk r = 5, ada dua gambaran graf rectilinier monoton r-teratur planar dengan 12 simpul dan 16 simpul.

Kata kunci: *Gambaran Monoton, Graf Planar, Graf Rectilinier, Graf Teratur.*

1. Introduction

The Pregolya River passes through a city once known as Königsberg. In the 1700s seven bridges were situated across this river in a manner similar to what you see in figure 1. The city's residents enjoyed strolling on these bridges, but, as hard as they tried, no resident of the city was ever able to walk a route that crossed each of these bridges exactly once. The Swiss mathematician Leonhard Euler learned of this frustrating phenomenon, and in 1736 he wrote an article about it. His work on the "Königsberg Bridge Problem" is considered by many to be the beginning of the field of graph theory **[1]**.

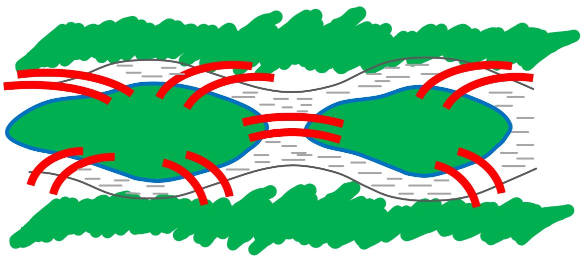


Figure 1. Illustration of The Königsberg Bridge

A graph (G) consists of nonempty set of points called *vertex/vertices* (also called *node/nodes*) and the set of lines or curves connecting two points (vertices) called *edge/edges*. The vertex set of a graph G is denoted by *V(G)* and the edge set is denoted by *E(G)*. Figure 2 shows a graph representation of the Königsberg Bridge. *V(G) = {A, B, C, D}*, representing the lands and *E(G) = {1, 2, 3, 4, 5, 6, 7}*, representing the bridges in Königsberg City.

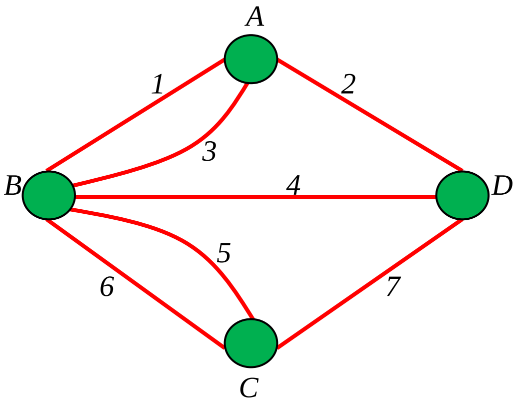


Figure 2. A Graph Representation of The Königsberg Bridge

A graph can be drawn in straight-line segment. If every edges of a graph drawn in straight-line segment then it is called *Rectilinear Graph*. The drawing of a graph without edges crossing is called *Planar Graph* and if it has edges crossing is called *Nonplanar Graph*.

A drawing of a graph is *x-monotone* if every edges intersects every vertical line at most once and every vertical line contains at most one vertex **[2]**.

A graph where each of its vertex has the same degree is called *Regular Graph*. If degree of each of vertex is *r* then it called *r-regular*. According to **[3]**, the number of edges in regular graphs is $\frac{nr}{2}$.

The idea of this thesis is to construct Rectilinear Monotone r-Regular Planar Graphs for:

1. $r = 3$
2. $r = 4$
3. $r = 5$

2. Graphs

The concepts and terminology of graphs theory are taken from **[4]**.

Definition 1 (Graphs)

A simple graph G is a finite nonempty set of objects called *vertices* together with a set of unordered pairs of distinct vertices of *G* called *edges*. The vertex set of *G* is denoted *V(G)* and the edge set is denoted *E(G)*.

Given two vertices, *u* and *v*, of graph *G*, the edge $e = (u,v)$ joins *u* and *v*. Common notation for the edge $e = (u,v)$ is *uv*. This more convenient notation is used when possible. If $e = uv$ is an edge of *G*, then *u* and *v* are called *adjacent vertices*, *u* and *e* are *incident* as are *v* and *e*. If e_1 and e_2 are distinct edges of *G* with a common vertex, it is said that e_1 and e_2 are *adjacent edges*.

Definition 2 (Order)

The *order* of graph *G* is the cardinality of the vertex set of *G*, commonly denoted *n(G)* or *n*.

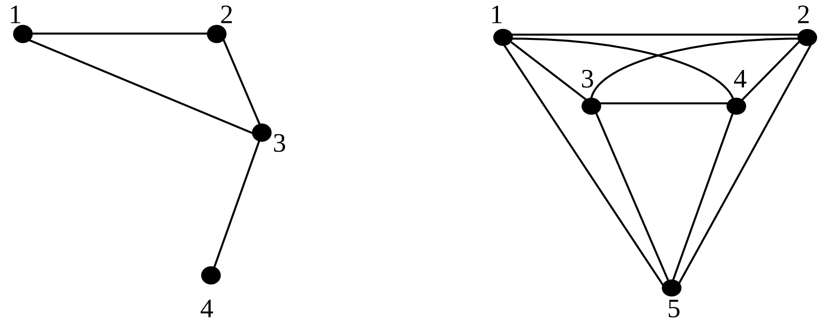
Definition 3 (Size)

The *size* of graph *G* is the cardinality of the edge set of *G*, commonly denoted *m(G)* or *m*.

Definition 4 (Degree)

The *degree* of a vertex *v* is the number of edges incident with *v*, commonly denoted *deg v*.

In figure 3 the above terms are illustrated for two graphs. The graph in (a) has order n = 4 and size m = 4. Vertices 1 and 2 each have degree 2, vertex 3 has degree 3, and vertex 4 has degree 1. Graph (b) has order n = 5, size m = 10, and all five vertices have degree 4.

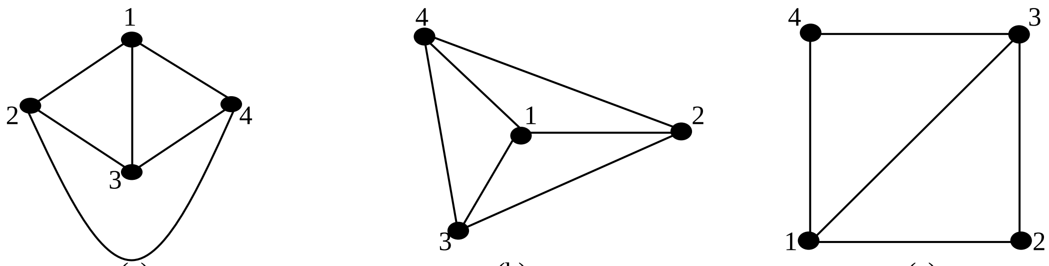


(a) (b)

Figure 3. Examples of Graphs

Definition 5 (Isomorphic)

A graph G is *isomorphic* to graph H if there exists a one-to-one mapping Φ, called an *isomorphism,* from V(G) onto V(H) such that Φ preserves adjacency; i.e. $uv \in E(G)$ *if and only if* $\Phi(u)\,\Phi(v) \in E(H)$.



(a) (b) (c)

Figure 4. Isomorphic and Nonisomorphic Graphs

The two graphs in figure 4 (a) and (b) are isomorphic, but graph (c) is not isomorphic to either (a) or (b).

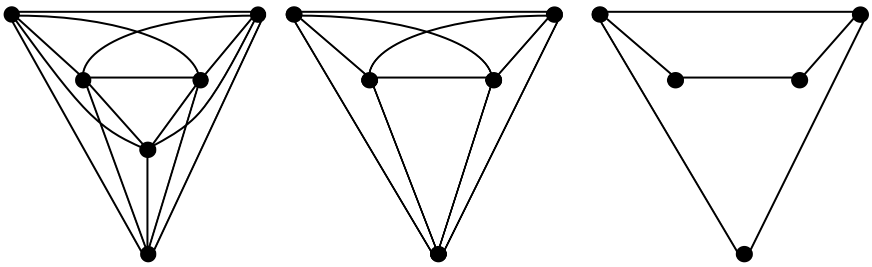
Definition 6 (Subgraphs)

A graph G is *subgraph* of graph H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

Definition 7 (Complete Graphs)

A graph G is *complete* if every two of its vertices are adjacent. The notation for the *complete graph* of order n is K_n.

In figure 5, graph (a) is a complete graph of order six. Graph (b) is a complete graph of order 5, and graph (c) is not complete. Graphs (b) and (c) are both subgraphs of graph (a).



(a) (b) (c)

Figure 5. A Complete Graph and Two Subgraphs

2.2 A Drawing of Graphs

According to **[5]**, a *drawing D(G)* of a graph *G* in a plane is a mapping of the graph to the plane in such a way that vertices are mapped to distinct points of the plane and edges are mapped into arcs terminating at the appropriate vertex points. All the drawing under consideration are assumed to be *simple*, i. e.:

i) no edge crosses itself;
ii) adjacent edges never cross;
iii) two non-adjacent edges cross at most once;
iv) no more than two edges cross at a point of the plane.

The illegal edges for a good graph as shown in figure 6 described by **[4]**.

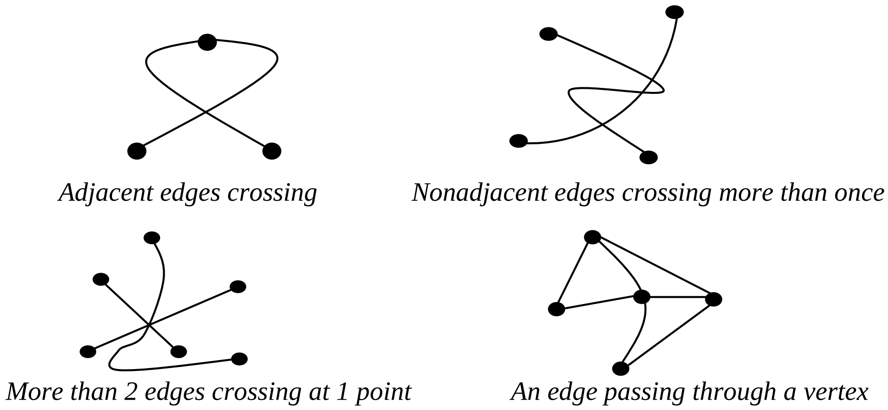


Figure 6. Illegal Edges for A Good Graph **[4]**

2.3 Rectilinear Drawings

In a rectilinear (or geometric) drawing of a graph G, the vertices of G are represented by points, and an edge joining two vertices is represented by the straight segment joining the corresponding two points. Edges are allowed to cross, but an edge cannot contain a vertex other than its endpoints **[6]**.

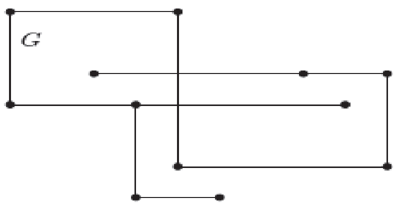


Figure 7. A Rectilinear Graph **[7]**

2.4 Monotone Drawings

A drawing of a graph is *x-monotone* if every edges intersects every vertical line at most once and every vertical line contains at most one vertex **[2]**.

2.5 Regular Graphs

A graph in which each vertex has the same degree is a *regular graph.* If each vertex has degree *r*, the graph is *regular of degree r or r-regular.* Of special importance are the cubic graphs, which are regular of degree 3. Note that the null graph N_n is regular of degree 0, the cycle graph C_n is regular of degree 2, and the complete graph K_n is regular of degree *n-1* **[8]**. According to **[3]**, the number of edges in regular graphs is $\frac{nr}{2}$.

(a) Graph N_n (b) Graph C_n

Figure 8. Examples of Regular Graphs **[8]**

Note that number of edges in regular graphs is integers. It means if r is an odd number then n is an even number. Let r and n are odd numbers, $r = 2p + 1$ and $n = 2q + 1$ for $p, q \in Z^+$, then $\frac{nr}{2} = \frac{(2q+1)(2p+1)}{2} = 2pq + p + q + \frac{1}{2} \notin Z^+$. Let n is an even number, $n = 2q$ for $q \in Z^+$, then $\frac{nr}{2} = \frac{(2q)(2p+1)}{2} = q(2p + 1) \in Z^+$. It satisfies for r is odd number then n should be even number.

2.6 Planarity and Planar Embedding

A *planar graph* is a graph that can be drawn in the plane without crossings – that is, so that no two edges intersect geometrically except at a vertex to which both are incident. Any such drawing is a *plane drawing* **[8]**.

A graph G is said to be *planar* if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. If G has no such representation, G is called *nonplanar*. A drawing of a planar graph G in the plane in which edges intersect only at vertices is called a *planar representation* (or a *planar embedding*) of G **[1]**.

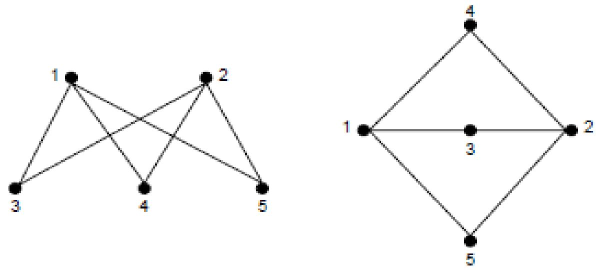
Definition 8 (Planar Graphs)

A graph is *planar* if it can be embedded in the plane **[4]**.

Definition 9 (Plane Graphs)

A planar graph that is embedded in the plane is called *a plane graph* **[4]**.

Figure 9 graph (a) is a planar graph, though as drawn it is not plane. The illustration in (b) is its plane representation.



(a) (b)

Figure 9. (a) A Planar and (b) Plane Graph **[4]**

Note that every subgraph of a planar graph is planar, and that every graph with a non-planar subgraph must be non-planar. It follows that any graph with $K_{3,3}$ or K_5 as a subgraph is non-planar **[8]**.

According to **[9]**, a connected part of a plane which does not contain any vertices and is surrounded by edges is called a *region* of a planar embedding. In addition, the part outside the embedding is considered as a region, known as the *exterior region* (when we draw a planar graph on a plane or on a sphere, it is just like any other region). The vertices surrounding a region s are called *boundary vertices* and the edges surrounding s are called *boundary edges*. Two regions are *adjacent* if they share a boundary edge. Note that a region can be adjacent to itself.

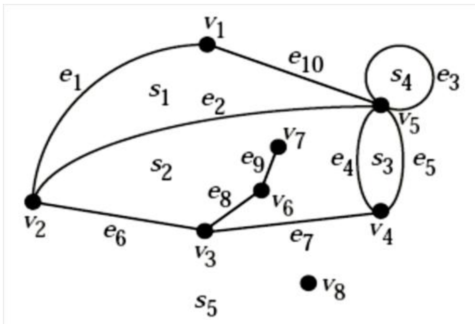


Figure 10. A Planar Embedding **[9]**

In figure 10 the regions are s_1, s_2, s_3, s_4 and s_5 (the exterior region) and their boundary vertices and edges as well as their adjacent regions are given in the table below:

Table 1. Regions, Boundaries, and Adjacents **[9]**

Region	Boundary Vertices	Boundary Edges	Adjacent Regions
s_1	v_1, v_5, v_2	e_1, e_{10}, e_2	s_2, s_5
s_2	v_2, v_5, v_4, v_3	$e_2, e_4, e_7, e_9, e_8, e_6$	s_1, s_2, s_3, s_5
s_3	v_4, v_5	e_4, e_5	s_2, s_5
s_4	v_5	e_3	s_5
s_5	$v_1, v_5, v_4, v_3, v_2, v_8$	$e_{10}, e_3, e_5, e_7, e_6, e_1$	s_1, s_2, s_3, s_4

There are theorems about planarity as the following below:

Theorem 1 (Planarity)

$K_{3,3}$ and K_5 are non-planar **[8]**.

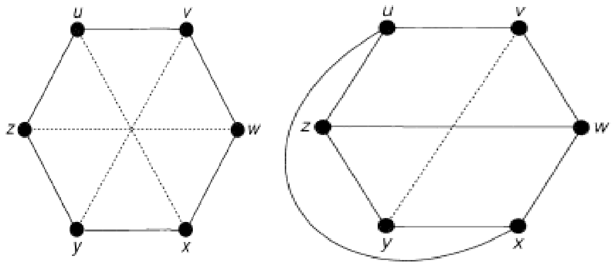


Figure 11. Graph $K_{3,3}$ **[8]**

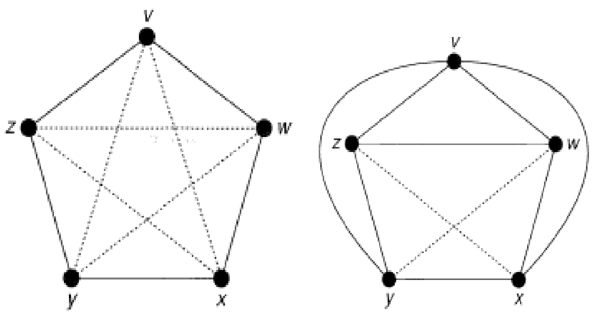


Figure 12. Graph K_5 **[8]**

Figure 11 and figure 12 show that the drawings of graph $K_{3,3}$ and graph K_5 has 1 crossing number.

Theorem 2 (Planarity)

Every simple planar graph contains a vertex of degree at most 5 **[8]**.

Theorem 3 (Kuratowski's Theorem)

A graph is planar if and only if none of its subgraphs can be transformed to K_5 or $K_{3,3}$ by contracting edges **[9]**.

Theorem 4 (Euler's Polyhedron Formula)

If a planar embedding of a connected graph G has n vertices, m edges and f regions, then $f + n = m + 2$ **[9]**.

3. Research Method

The methods using in this research are study literature from books and journals. The constructions of rectilinear monotone regular planar graphs are developed by learn the characteristics of each kind of graph.

4. Results And Discussions

Rectilinear Monotone r-Regular Planar Graphs is a simple connected graph that consists of vertices with same degree and horizontal or diagonal straight edges without vertical edges and edges crossing.

4.1 Rectilinear Monotone 3-Regular Planar Graphs

Let graph A_n is a simple connected graph. There are infinite family of rectilinear monotone 3-regular planar graphs with general form:

$A_n = \{(V_n, E_n) | \, n \in \mathbb{N} \wedge n \geq 3\}$, where

$V_n = \{p_1, p_2, \ldots, p_{n-1}, p_n, q_1, q_2, \ldots, q_{n-1}, q_n\}$

$E_n = \{p_k q_k | 1 \leq k \leq n\} \cup \{p_k p_{k+1} | 1 \leq k \leq n-1\} \cup \{q_k q_{k+1} | 1 \leq k \leq n-1\} \cup \{p_1 p_n\} \cup \{q_1 q_n\}$

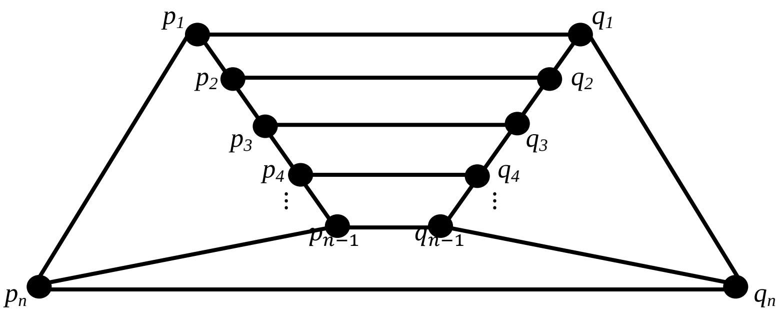


Figure 13. Graph A_n

Figure 13 shows that the vertices of graph A_n are represented by points and every vertex has degree 3. It means graph A_n is 3-regular. Each of its edge joining two vertices is represented by straight segment and it intersects every vertical line at most a point. Then graph A_n is rectilinear and it is also x-monotone. Graph A_n is planar embedding.

To show that there are infinite family of graph $A_n = \{(V_n, E_n) | \, n \in \mathbb{N} \wedge n \geq 3\}$, we must prove for $n = 3$ there is graph A_3.

4.1.1 Example of Graph A_3

$A_3 = \{(V_3, E_3)\}$

$V_3 = \{p_1, p_2, p_3, q_1, q_2, q_3\}$

$E_3 = \{p_k q_k | 1 \leq k \leq 3\} \cup \{p_k p_{k+1} | 1 \leq k \leq 2\} \cup \{q_k q_{k+1} | 1 \leq k \leq 2\} \cup \{p_1 p_3\} \cup \{q_1 q_3\}$

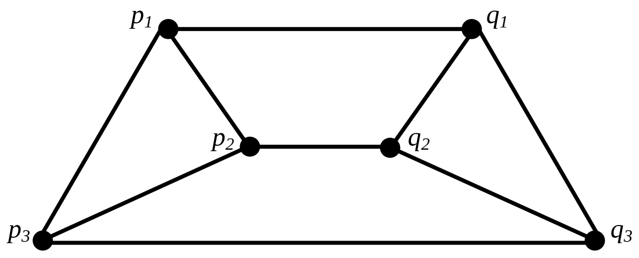


Figure 14. Graph A_3

Then, assume that there is graph A_n for all $n \in \mathbb{N}$. It will be proved that there is graph A_{n+1}:

1) p_1p_n and q_1q_n deleted → rerouted p_n and q_n
2) $p_{n+1}q_{n+1}$ added, p_np_{n+1} and q_nq_{n+1} added, p_1p_{n+1} and q_1q_{n+1} added

4.1.2 Example of Graph A_4

$A_4 = \{(V_4, E_4)\}$
$V_4 = \{p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4\}$
$E_4 = \{p_kq_k | 1 \leq k \leq 4\} \cup \{p_kp_{k+1} | 1 \leq k \leq 3\} \cup \{q_kq_{k+1} | 1 \leq k \leq 3\} \cup \{p_1p_4\} \cup \{q_1q_4\}$

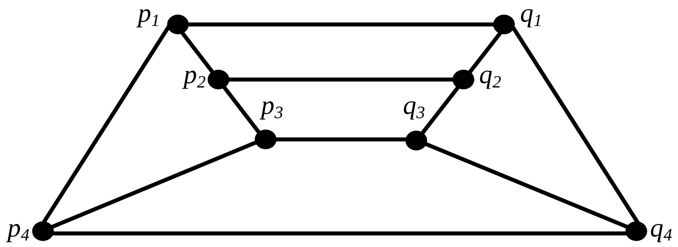


Figure 15. Graph A_4

4.2 Rectilinear Monotone 4-Regular Planar Graphs

Let graph R_n is a simple connected graph. There are infinite family of rectilinear monotone 4-regular planar graphs with general form:

$R_n = \{(V_n, E_n) \mid n \in \mathbb{N} \wedge n \geq 3\}$, where
$V_n = \{p_1, \dots, p_{n-1}, p_n, q_1, \dots, q_{n-1}, r_1, \dots, r_{n-1}, r_n\}$
$E_n = \{p_kp_{k+1} \mid 1 \leq k \leq n-1\} \cup \{p_1p_n\} \cup \{p_kq_k \mid 1 \leq k \leq n-1\} \cup \{r_kr_{k+1} \mid 1 \leq k \leq n-1\} \cup \{r_1r_n\} \cup \{r_kq_k \mid 1 \leq k \leq n-1\} \cup \{p_1r_1\} \cup \{p_nr_n\} \cup \{q_kp_{k+1} \mid 1 \leq k \leq n-1\} \cup \{q_kr_{k+1} \mid 1 \leq k \leq n-1\}$

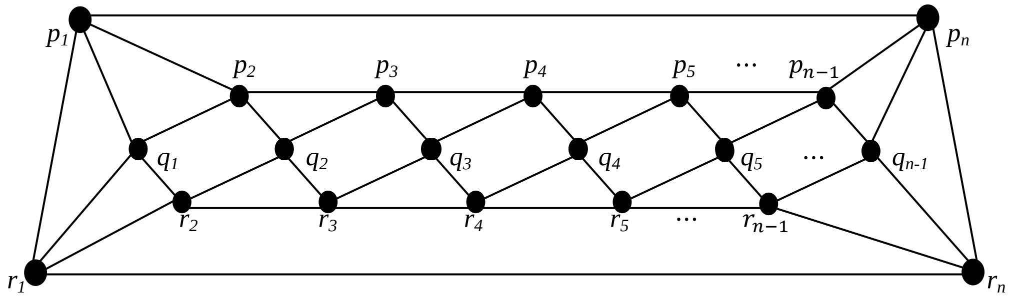


Figure 17. Graph R_n

Figure 17 shows that the vertices of graph R_n are represented by points and every vertex has degree 4. It means graph R_n is 4-regular. Each of its edge joining two vertices is represented by straight segment and it intersects every vertical line at most one point. Then graph R_n is rectilinear and it is also x-monotone. Graph R_n is planar embedding.

To show that there are infinite family of graph $R_n = \{(V_n, E_n) | n \in \mathbb{N} \wedge n \geq 3\}$, we must prove for $n = 3$ there is graph R_3.

4.2.1 Example of Graph R_3

$R_3 = \{(V_3, E_3)\}$
$V_3 = \{p_1, p_2, p_3, q_1, q_2, r_1, r_2, r_3\}$
$E_3 = \{p_k p_{k+1} \mid 1 \leq k \leq 2\} \cup \{p_1 p_3\} \cup \{p_k q_k \mid 1 \leq k \leq 2\} \cup \{r_k r_{k+1} \mid 1 \leq k \leq 2\} \cup \{r_1 r_3\} \cup \{r_k q_k \mid 1 \leq k \leq 2\} \cup \{p_1 r_1\} \cup \{p_3 r_3\} \cup \{q_k p_{k+1} \mid 1 \leq k \leq 2\} \cup \{q_k r_{k+1} \mid 1 \leq k \leq 2\}$

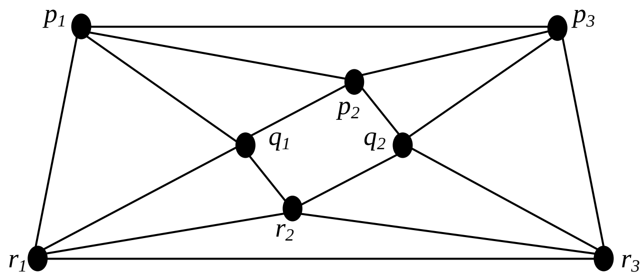


Figure 18. Graph R_3

Then, assume that there is graph R_n for all $n \in \mathbb{N}$. It will be proved that there is graph R_{n+1}:

1) $p_1 p_n$, $r_1 r_n$ and $p_n r_n$ deleted → rerouted p_n and r_n
2) $p_{n+1} r_{n+1}$, $q_n p_{n+1}$ and $q_n r_{n+1}$ added, $p_n p_{n+1}$ and $r_n r_{n+1}$ added, $p_1 p_{n+1}$ and $r_1 r_{n+1}$ added

4.2.2 Example of Graph R_4

$R_4 = \{(V_4, E_4)\}$
$V_4 = \{p_1, p_2, p_3, p_4, q_1, q_2, q_3, r_1, r_2, r_3, r_4\}$
$E_4 = \{p_k p_{k+1} \mid 1 \leq k \leq 3\} \cup \{p_1 p_4\} \cup \{p_k q_k \mid 1 \leq k \leq 3\} \cup \{r_k r_{k+1} \mid 1 \leq k \leq 3\} \cup \{r_1 r_4\} \cup \{r_k q_k \mid 1 \leq k \leq 3\} \cup \{p_1 r_1\} \cup \{p_4 r_4\} \cup \{q_k p_{k+1} \mid 1 \leq k \leq 3\} \cup \{q_k r_{k+1} \mid 1 \leq k \leq 3\}$

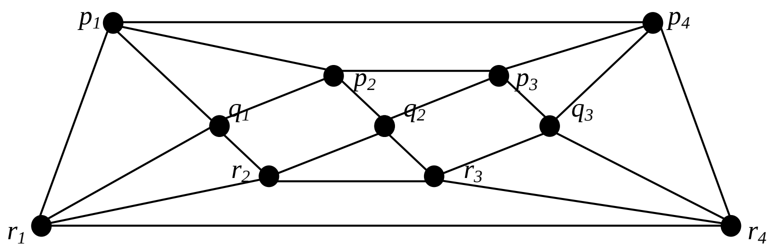


Figure 19. Graph R_4

4.3 Rectilinear Monotone 5-Regular Planar Graphs

This section shows two different drawings of rectilinear monotone 5-regular planar graphs. The first drawing consists of 12 vertices and 30 edges. The second consists of 16 vertices and 40 edges. Note that first drawing is irregular.

4.3.1 Rectilinear Monotone 5-Regular Planar Graphs with 12 Vertices

Let graph W is a simple connected graph. There is a drawing of rectilinear monotone 5-regular planar graph with 12 vertices and 30 edges. The graph $W = \{V, E\}$ consists of:
$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_{11}, v_2v_3, v_2v_5, v_2v_7, v_2v_{11}, v_3v_4, v_3v_5, v_3v_6, v_4v_6, v_4v_9, v_4v_{12}, v_5v_6, v_5v_7, v_5v_8, v_6v_8, v_6v_9, v_7v_8, v_7v_{10}, v_7v_{11}, v_8v_9, v_8v_{10}, v_9v_{10}, v_9v_{12}, v_{10}v_{11}, v_{10}v_{12}, v_{11}v_{12}, v_{12}v_1\}$

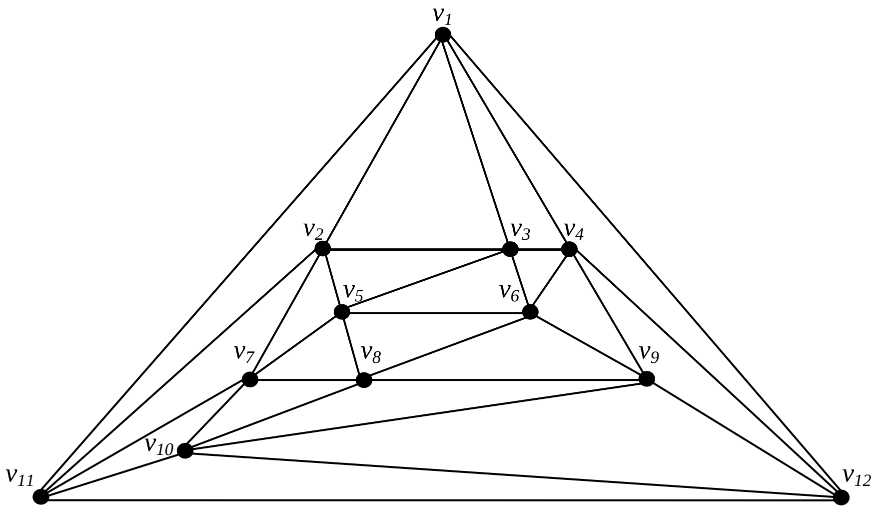


Figure 21. Graph W

4.3.2 Rectilinear Monotone 5-Regular Planar Graphs with 16 Vertices

Let graph F is a simple connected graph. There is a drawing of rectilinear monotone 5-regular planar graphs with 16 vertices and 40 edges. The graph $F = \{V, E\}$ consists of:

$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$

$E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_7, v_1v_{15}, v_2v_3, v_2v_6, v_2v_{10}, v_2v_{16}, v_3v_4, v_3v_5, v_3v_6, v_4v_5, v_4v_7, v_4v_8, v_5v_6, v_5v_8, v_5v_9, v_6v_9, v_6v_{10}, v_7v_8, v_7v_{11}, v_7v_{15}, v_8v_{11}, v_8v_{12}, v_9v_{10}, v_9v_{12}, v_9v_{13}, v_{10}v_{13}, v_{10}v_{16}, v_{11}v_{12}, v_{11}v_{14}, v_{11}v_{15}, v_{12}v_{13}, v_{12}v_{14}, v_{13}v_{14}, v_{13}v_{16}, v_{14}v_{15}, v_{14}v_{16}, v_{15}v_{16}\}$

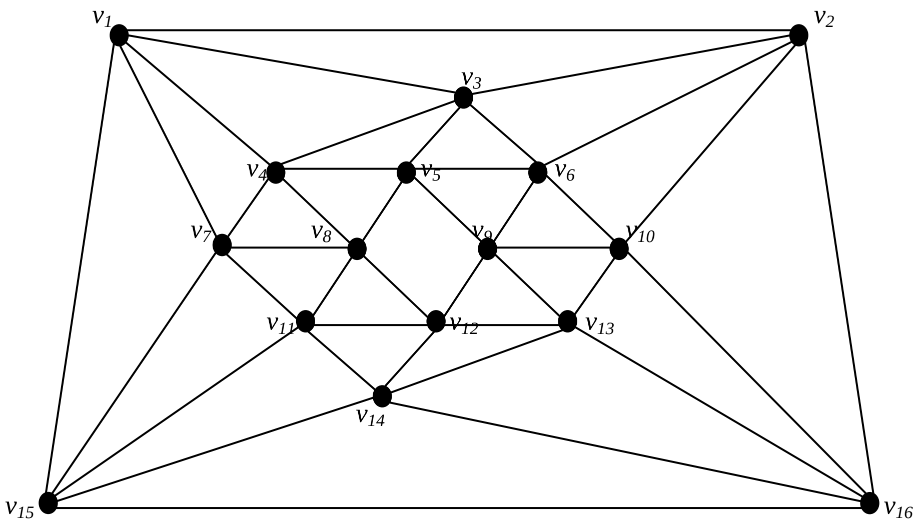


Figure 22. Graph F

5. Conclusions And Suggestion

5.1 Conclusions

Based on the results and discussions this research has some following conclusions:

1. There are infinite family of rectilinear monotone 3-regular planar graphs with general form:
   $A_n = \{(V_n, E_n) | \ n \in \mathbb{N} \wedge n \geq 3\}$, where
   $V_n = \{p_1, p_2, \dots, p_{n-1}, p_n, q_1, q_2, \dots, q_{n-1}, q_n\}$
   $E_n = \{p_kq_k | 1 \leq k \leq n\} \cup \{p_kp_{k+1} | 1 \leq k \leq n-1\} \cup \{q_kq_{k+1} | 1 \leq k \leq n-1\} \cup \{p_1p_n\} \cup \{q_1q_n\}$

2. There are infinite family of rectilinear monotone 4-regular planar graphs with general form:
$R_n = \{(V_n, E_n) \mid n \in \mathbb{N} \wedge n \geq 3\}$, where
$V_n = \{p_1, \ldots, p_{n-1}, p_n, q_1, \ldots, q_{n-1}, r_1, \ldots, r_{n-1}, r_n\}$
$E_n = \{p_k p_{k+1} \mid 1 \leq k \leq n-1\} \cup \{p_1 p_n\} \cup \{p_k q_k \mid 1 \leq k \leq n-1\} \cup$
$\{r_k r_{k+1} \mid 1 \leq k \leq n-1\} \cup \{r_1 r_n\} \cup \{r_k q_k \mid 1 \leq k \leq n-1\} \cup \{p_1 r_1\} \cup \{p_n r_n\} \cup$
$\{q_k p_{k+1} \mid 1 \leq k \leq n-1\} \cup \{q_k r_{k+1} \mid 1 \leq k \leq n-1\}$
3. There are two drawings of rectilinear monotone 5-regular planar graphs:
a) Graph W consists of 12 vertices and 30 edges
b) Graph F consists of 16 vertices and 40 edges

5.2 Suggestion

This thesis only shows two drawings of rectilinear monotone 5-regular planar graphs. It is suggested try to find the infinite family of rectilinear monotone 5-regular planar graphs.

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