Perfect One-Factorization Conjecture

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Abstract

Perfect one-factorization of the complete graph K_{2n} for all n greater and equal to 2 is conjectured. Nevertheless some families of complete graphs were found to have perfect one-factorization. This paper will show some of the perfect one-factorization results in some families of complete graph as well as some result in application.

Keywords: complete graph, one-factorization

Dugaan Perfect One-Factorization

Abstrak

Perfect one-factorization pada graph lengkap K_{2n} untuk semua n lebih dari dan sama dengan 2 masih diduga. Namun beberapa graph lengkap dibuktikan memiliki perfect one-factorization. Tulisan ini akan menunjukkan beberapa perfect one-factorization yang ada pada beberapa graph dan juga beberapa hasil dalam aplikasinya.

Kata kunci: *graph lengkap, one-factorization*

1. Introduction

The existence of a perfect one-factorization of the complete graph K_{2n} for all $n \geq 2$ is conjectured. This has been suggested by Kotzig in 1964 in his paper "Hamiltonian Graphs and Hamiltonian Circuits" [1].

Three families of perfect one-factorization are known to be exist. The first family was proven by Kotzig himself. In his paper [1], Kotzig proved that there exists a perfect one-factorization of the complete graph K_{p+1} for all odd primes p. The second family was constructed by Anderson in 1973 [2], i.e. the perfect one-factorization for complete graph K_{2p} exists for all odd prime p. This was also constructed independently by Nakamura in 1975 [3]. Nevertheless, these two results were proven to be isomorphic by Kobayashi [4] in 1989. Long after that, in 2006, a new family which is the third family was shown in [5]. Bryant, Maenhaut and Wanless constructed a perfect one-factorization of K_{p+1}, for each prime $p \geq 11$ which is not isomorphic to complete graph in the first family. These three families have become reference for the studies in perfect one-factorization, especially for the work of proving the existence of perfect one-factorization for all complete graph K_{2n}.

This essay outlines some studies in perfect one-factorization conjecture as well as the result of some complete graphs that have been shown to have perfect one-factorization and some brief applications of perfect one-factorization.

2. Perfect One-Factorization

For any graph *G*, a ***factor*** or a ***spanning subgraph*** of *G* is a subgraph with vertex-set *V*(*G*). A ***factorization*** of *G* is a set of factors of *G* that are pairwise edge-disjoint (no two have a common edge) and whose union is all of *G*. In particular, a ***one-factor*** is a factor that is a regular graph of degree 1, in other words, a *one-factor* is a set of pairwise disjoint edges of *G* that between them contain every vertex. A ***one-factorization*** of *G* is a decomposition of the edge-set of *G* into edge disjoint one-factors.

Example 1. A graph with one-factor and one-factorization in Desargues graph

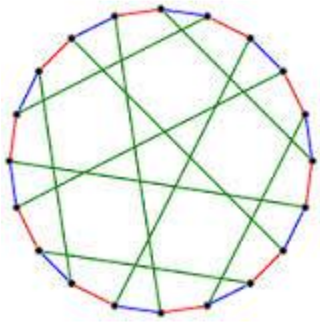

Figure 1. Desargues graph

There are three one-factors in the picture edge-colored by three different colors (red, blue, green). The union of the three factors forms the graph, thus this is an example of a graph that has one-factorization.

A *perfect one-factorization* of *G* is a one-factorization in which the union of any pair of one-factors is a Hamiltonian cycle of *G*. Consider the graph in the example above, the union of the red one-factor and the blue-one factor induces a Hamiltonian cycle. Furthermore, it is also can be seen that both the union of the red and green and the blue and green induce Hamiltonian cycles in the graph. Therefore, that graph is also an example of a graph with perfect one-factorization.

3. Some Studies of Perfect One-Factorization Conjecture

3.1. Perfect Pair of One-Factorization and Perfect Near One-Factorization

In his paper in 1989, Wagner [6] proved the lower bound of maximum perfect pairs of one-factorization in a complete graph. This was shown by first considering a graph *G* that has a one-factorization $\mathscr{F}$ with set of one-factors $\mathscr{F} = \{F_1, \dots, F_d\}$. The *perfect pair* of $\mathscr{F}$ is a pair $\{F_k, F_l\}$ such that $F_k \cup F_l$ induces a Hamiltonian cycle in *G*.

Let $c(\mathscr{F})$ to be the number of perfect pairs of $\mathscr{F}$, and $c(G)$ to be the maximum $c(\mathscr{F})$ over all one-factorizations $\mathscr{F}$ of *G*. Since the conjecture is the existence of a perfect one-factorization of the complete graph K_{2n} for all $n \geq 2$, it can be assumed that a one-factorization of K_{2n} exists in which every pair is perfect. Wallis in [7] proved in theorem 6.3. that if v is even, then K_v can be factored into $\frac{v}{2} - 1$ Hamilton cycles and a one factor. This implies that there can be $2n - 1$ one-factors in a K_{2n}. Thus if K_{2n} has a one-factorization $\mathscr{F}$ with the set of one-factors $\mathscr{F} = \{F_1, \dots, F_d\}$, then $d = 2n - 1$. Then, the perfect one-factorization conjecture for $n \geq 2$ satisfies

$$c(K_{2n}) = \binom{2n-1}{2}.$$

The related problem in this case is to find a set of $\binom{2n-1}{2}$ Hamilton cycles in K_{2n} such that each edge in K_{2n} is on exactly one of the cycles.

Furthermore, in this paper *perfect near-one-factorization* is also considered to be useful in determining the conjecture. It is well known that K_{2n} has a perfect one-factorization if and only if K_{2n-1} has a perfect near one-factorization. Recall that a *near-one-factor* of $G = (V, E)$ is a one-factor of $G \setminus v$ for some $v \in V$ and a *near-one-factorization* of G is a partition $\mathscr{F} = \{F_1, \dots, F_s\}$ of E into *near-one-factors*. A pair $\{F_k, F_l\}$ in $\mathscr{F}$ is a *perfect pair* if $F_k \cup F_l$ induces a Hamiltonian path in *G*. A near-one-factorization is *perfect* if and only if all of its pairs are perfect.

For odd $2n - 1$, $c(K_{2n}) = \binom{2n-1}{2}$ if n is prime or $2n - 1$ is prime. This leads to the families of complete graph with perfect one-factorization. Based on these, the lower bound for the maximum of perfect pair in complete graph of order $2n$ can be obtained as:

$$c(K_{2n}) \geq (2n-1).\varphi(2n-1)/2, \text{ where } \varphi \text{ is the Euler totient function}$$

Moreover, by considering this inequality, another inequality can be obtained as follows:

If m and n are odd and coprime then

$$c(K_{mn}) \geq 2.c(K_m).c(K_n)$$

Since mn is odd which means $mn + 1$ is even, then regarding the *perfect near-one-factorization* properties, one can obtain:

If m and n are odd and coprime then
$$c(K_{mn+1}) \geq 2.\, c(K_{m+1}).\, c(K_{n+1})$$

These inequalities can lead to assumption that if the maximum number of perfect pairs in one-factorization set of complete graph of even number can be obtained, at least an estimated number, then there may exist perfect pairs for all one-factorizations of complete graph of even number, K_{2n} for all $n \geq 2$. Therefore, further study for the conjecture which is based on these results can be conducted.

3.2. $2^{\alpha-1}$-*quotient Starters In Finite Fields*

Dinitz and Stinson [8] constructed seven examples of perfect one-factorizations by using two- and four-quotient starters in finite fields. A *starter* in an additive abelian group G of order $2n-1$ is a set $S = \{\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_{n-1}, y_{n-1}\}\}$ such that every non-zero element of G occurs as

(1) an element in exactly one pair of S, and

(2) a difference of exactly one pair of S.

For example, $\{\{1,6\},\ \{2,5\},\ \{3,4\}\}$ is a starter in Z_7.

Let $S^* = S \cup \{0, \infty\}$. For any $g \in G$, define
$$S^* + g = \{\{x_1 + g,\ y_1 + g\}, \{x_2 + g,\ y_2 + g\}, \dots, \{x_{n-1} + g,\ y_{n-1} + g\}\},$$
where $\infty + g = g + \infty = \infty$ for all $g \in G$

Then, it is clear that $F = \{S^* + g\}$ for all $g \in G$ is a one-factorization of K_{2n}. To check the *perfect* properties of this, instead of checking all $\binom{2n-1}{2}$ pairs of one-factors, choose $n-1$ nonzero group elements $g_1, g_2, \dots, g_{n-1}$ such that no two of them sum to zero. Then, F is perfect if and only if $S^* \cup (S^* + g_i)$ is a Hamiltonian cycle for $1 \leq i \leq n-1$.

For example $S^* = \{\{1,6\},\ \{2,5\},\ \{3,4\},\ \{0,\infty\}\}$.

Consider $G = \{0, 1, 2, 3, 4, 5, 6\}$

Then for all $g \in G$, $S^* + g$ will give an orbit in K_8 as follows:

$$\begin{aligned} S^* + g = \{&\{1,6\},\{2,5\},\{3,4\},\{0,\infty\},\\ &\{2,0\},\{3,6\},\{4,5\},\{1,\infty\},\\ &\{3,1\},\{4,0\},\{5,6\},\{2,\infty\},\\ &\{4,2\},\{5,1\},\{6,0\},\{3,\infty\},\\ &\{5,3\},\{6,2\},\{0,1\},\{4,\infty\},\\ &\{6,4\},\{0,3\},\{1,2\},\{5,\infty\},\\ &\{0,5\},\{1,4\},\{2,3\},\{6,\infty\}\} \end{aligned}$$

$S^* + g$ is a one-factorization of K_8 with vertex set $V = \{0, 1, 2, 3, 4, 5, 6, \infty\}$.

Check the perfect properties as follows,

Suppose two distinct groups in which no two elements on each of them sum to zero.

Consider $G_1 = \{1, 2, 3\}$ and $G_2 = \{4, 5, 6\}$

then it is clear that for each $g_i \in G_1$ or G_2, $S^* \cup (S^* + g_i)$ is a Hamiltonian cycle. Thus, $F = \{S^* + g\}$ is perfect.

Suppose $q = 2^\alpha t + 1$ is an odd prime power, where t is odd.

Let ω be a primitive element in Galois field $GF(q)$, and let C_0 be the unique subgroup of G^* of order t and index 2^α, where G^* denotes the multiplicative group $G \setminus \{0\}$.

Denote the coset of C_0 by $C_i, 0 \leq i \leq 2^\alpha - 1$, where $C_i = \omega^i C_0$.

For example, let us consider $GF(13)$.

Since $13 = 2^2.3 + 1$, the sub-group C_i is of order 3 and there are $2^2 = 4$ subgroups

In multiplicative group $G\setminus\{0\}$, the set of coprime number to 13 is
$$G^* = \{1, 2, 3, \dots, 12\}$$

Choose a cyclic subgroup generated by 3, i.e. $3^0 = 1$, $3^1 = 3$, and $3^2 = 9$, then

$C_0 = \{1, 3, 9\}$, and the other cosets can be derived as follows:

$C_1 = \{2, 6, 5\}$, $C_2 = \{4, 12, 10\}$, and $C_3 = \{7, 8, 11\}$.

A starter S in $GF(q)$ is said to be a $2^{\alpha-1}$-quotient coset starter if the following property is satisfied:

for all pairs $\{x,y\},\{x',y'\} \in S$ if $x,x' \in C_i$ for some i, then $\frac{y}{x} = \frac{y\prime}{x\prime}$

For example, in polynomial form, consider $q = 25 = 5^2 = 2^3 8 + 1$. $GF(25)$ can be constructed from the irreducible polynomial $x^2 + x + 2$ over $GF(5)$. Then x is a primitive element. By calculating a subgroup with a generator, it can be found that $C_0 = (x^1, x^{13}, x^{10}, x^{14})$. This can be a starter in form $S(x^1, x^{13}, x^{10}, x^{14})$ and it is clear that it is a 4-quotient coset starter.

By using this fact, Dinitz and Stinson proved that a starter S, where S is a $2^{\alpha-1}$-quotient coset starter in $GF(q)$, generates a perfect one-factorization if and only if $S^* \cup (S^* + g_i)$ is a Hamiltonian cycle for $1 \leq i \leq 2^{\alpha-1} - 1$.

For example consider again the previous example for $q = 25$. The starter $S(x^1, x^{13}, x^{10}, x^{14})$ in $GF(25)$ generates perfect one-factorization. This can be done by checking that both $S^* \cup (S^* + x)$ and $S^* \cup (S^* + x^2)$ induce Hamiltonian cycle.

By using this method, perfect one-factorization of complete graph K_{2n} can be obtained.

For example, consider a complete graph of order 126, K_{126}.

Then take $q = 125 = 5^3 = 2^2 31 + 1$.

Construct $GF(5^3)$ from the polynomial $x^3 + x^2 + 2$, which is irreducible over Z_5. Then, x is a generator. $S(x^9, x^{41}) = S(2x^2 + 4x + 4, x^2 + 2x + 3)$ generates a perfect one-factorisation of K_{126}.

4. Uniform One-Factorization

Another approach to perfect one-factorization is by considering the *uniform* of one-factorization. This was used by Dinitz and Dukes in [9] regarding the cycle structure of one-factorization.

A one factorization $\{F_1, \dots, F_{2n-1}\}$ of K_{2n} is *uniform* if the graphs with edge sets $F_i \cup F_j$ are all isomorphic for $i \neq j$. Since the union of two one-factors is a 2-regular graph, then it is isomorphic to a disjoint union of even cycles. If $F_i \cup F_j$ is isomorphic to the disjoint union of cycles of lengths $k_1, \dots, k_r$, then the multiset $T = (k_1, \dots, k_r)$ is said to be the *type* of a uniform one-factorization. In other words, a uniform one-factorization "contains a k-cycle" if k occurs in its type. A one-factorization of K_{2n} is perfect if it is uniform of type $2n$.

In this paper, Dinitz and Dukes show that for each even $k > 4$ and any positive integer N there exists a uniform one-factorization in some large enough complete graph containing at least N number of k-cycles. This means that, with the knowledge of $2n$-cycle in a complete graph, there can be a possibility for a large complete graph of order $2n$ for all n positive integer to have perfect one-factorizations.

5. Some Constructed Results

1974, Anderson and Morse [10] constructed perfect one-factorizations of K_{16} and K_{28}

1988, Seah and Stinson [11] constructed perfect one-factorizations of K_{36}

1989, Seah and Stinson [12] constructed perfect one-factorizations of K_{40}

1987, Ihrig, Seah, and Stinson [13] constructed perfect one-factorizations of K_{50}

2009, Wolfe constructed perfect one-factorizations of K_{52}

1988, Kobayashi and Kiyasu-Zen'iti [14] constructed perfect one-factorizations of K_{1332} and K_{6860}

By using 2- and 4-quotient starters in finite fields, Dinitz and Stinson [9] constructed K_{126}, K_{170}, K_{730}, K_{170}, K_{1370}, K_{1850}, K_{2198}, and K_{3126}.

Wanless in webpage [15] listed the known existence results of a perfect one-factorization of K_{n+1} exists if n is odd and

- $n < 50$
- $n = p$ for some prime p (two non-isomorphic constructions known),
- $n = 2p - 1$ for some prime p
- $n = p^2$ for some prime $p < 280$, such that $p = 3$ or $5 \bmod 8$
- $n = p^2$ for some prime $p < 30$, such that $p = 7 \bmod 8$
- $n = p^3$ for some prime $p < 300$, such that $p = 3 \bmod 4$

- $n = p^3$ for some prime $p < 150$, such that $p = 5 \bmod 8$
- $n = p^4$ for some prime $p < 6$
- $n = p^5$ for some prime $p < 25$ except possibly $p = 17$
- $n = p^6$ for some prime $p < 6$
- $n = 5^7$

6. Some Applications

6.1. Shortening array codes

Array codes are erasure-correcting codes represented by an array bits. Erasures correspond to the loss of columns. A two erasure correcting array code, for example, is capable of recovering any two lost columns.

For example, a simple array codes
A simple two-erasure correcting array code of length four is shown in table 1.

Table 1. Array code

a	*b*	*c*	*d*
b+*c*	*c*+*d*	*d*+*a*	*a*+*b*

The first row consists of four information bits *a*, *b*, *c*, and *d*.
The second row contains four parity bits.
The '+' sign indicates bitwise exclusive-OR, so that $x + x = 0$.
Suppose, for example, that columns three and four are lost (as shown in Table 2).

Table 2. Lost array code

a	*b*		
b+*c*	*c*+*d*		

c can be recovered by adding $b + c$ to *b*: $c = (b + c) + b$
d can be recovered by using $c + d$ and *c*: $d = (c + d) + c$
The *B-Code* is a two-erasure correcting array code of length $2n$, represented by a n by $2n$ array. It can recover any two out of $2n$ lost columns. The construction of the B-Code is based on the perfect one-factorization of the complete graph, K_{2n}.

In [16], Bohossian and Bruck used the constructed perfect one-factorization to shorten arbitrary array codes. Moreover, these codes can be used to derive a new family of the conjecture. The derivation they used is as follows:
(1) Find an existed perfect one-factorization of K_{2n},
(2) Construct extended B-Code of length $2n - 1$, B_{2n-1}, from K_{2n} in (1)
(3) Generalize X-Code of length n, $\tilde{X}_n$ by shortening new construction from B_{2n-1},
(4) Construct extended B-Code of length n, B_n, from $\tilde{X}_n$ by separation
(5) Construct a new perfect one-factorization of K_{n+1} from B_n.

Above derivation can be done in an example as follows:
Consider K_{10} which is known to have a perfect one-factorization with one-factors that can be listed as shown in Table 3.

Table 3. One-factors of a perfect one-factorization

f_5	f_2	f_4	f_6	f_8	f_1	f_3	f_7	f_9
(0, 5)	(1, 6)	(2, 7)	(3, 8)	(4, 9)	(0, 1)	(0, 3)	(0, 7)	(0, 9)
(9, 1)	(0, 2)	(1, 3)	(2, 4)	(3, 5)	(2, 3)	(2, 5)	(2, 9)	(2, 1)
(8, 2)	(9, 3)	(0, 4)	(1, 5)	(2, 6)	(4, 5)	(4, 7)	(4, 1)	(4, 3)
(7, 3)	(8, 4)	(9, 5)	(0, 6)	(1, 7)	(6, 7)	(6, 9)	(6, 3)	(6, 5)
(6, 4)	(7, 5)	(8, 6)	(9, 7)	(0, 8)	(8, 9)	(8, 1)	(8, 5)	(8, 7)

Construct B_8 by deleting f_5 as well as vertices 0 and 5 and all edges connected to them.

Table 4. B_8

(1, 6)	(2, 7)	(3, 8)	(4, 9)	(2, 3)	(4, 7)	(2, 9)	(2, 1)
(9, 3)	(1, 3)	(2, 4)	(2, 6)	(6, 7)	(6, 9)	(4, 1)	(4, 3)
(8, 4)	(8, 6)	(9, 7)	(1, 7)	(8, 9)	(8, 1)	(6, 3)	(8, 7)
p_2	p_4	p_6	p_8	p_1	p_3	p_7	p_9

In this case, the edges are considered as information bits and p_i, for $1 \leq i \leq 9$ are considered as parity bits.

Construct extended B-Code of size 9, B_9, from B_8 by adding a column of information bits. Those are the bits corresponding to the edges of f_5 which was deleted in construction of B_8.

Table 5. B_9

(1, 9)	(1, 6)	(2, 7)	(3, 8)	(4, 9)	(2, 3)	(4, 7)	(2, 9)	(2, 1)
(2, 8)	(9, 3)	(1, 3)	(2, 4)	(2, 6)	(6, 7)	(6, 9)	(4, 1)	(4, 3)
(3, 7)	(8, 4)	(8, 6)	(9, 7)	(1, 7)	(8, 9)	(8, 1)	(6, 3)	(8, 7)
(4, 6)	p_2	p_4	p_6	p_8	p_1	p_3	p_7	p_9

The shortening from B_9 into $\tilde{X}_5$ can be obtained by first construct the shortening from B_8 into $\tilde{X}_4$. Set all information bits in the last 4 columns of array representation in B_8 to zero.

Table 6. $\tilde{X}_4$

(1, 6)	(2, 7)	(3, 8)	(4, 9)	0	0	0	0
(9, 3)	(1, 3)	(2, 4)	(2, 6)	0	0	0	0
(8, 4)	(8, 6)	(9, 7)	(1, 7)	0	0	0	0
p_2	p_4	p_6	p_8	p_1	p_3	p_7	p_9

The parities then can be written as:
$p_1 = (1, 6) + (1, 3) + (1, 7)$
$p_3 = (9, 3) + (1, 3) + (3, 8)$
$p_7 = (2, 7) + (9, 7) + (1, 7)$
$p_9 = (9, 3) + (9, 7) + (4, 9)$
which then can be rewritten as
$(1, 6) = p_1 + (1, 3) + (1, 7)$
$(3, 8) = (9, 3) + (1, 3) + p_3$
$(2, 7) = p_7 + (9, 7) + (1, 7)$
$(4, 9) = (9, 3) + (9, 7) + p_9$

Rename (1, 6), (3, 8), (2, 7), (4, 9) as parities and p_1, p_3, p_7, p_9 as information bits, and set the information bits to zero as shown in table 7.

Table 7. Zero information bits

p_1	p_7	p_3	p_9	0	0	0	0
(9, 3)	(1, 3)	(2, 4)	(2, 6)	0	0	0	0
(8, 4)	(8, 6)	(9, 7)	(1, 7)	0	0	0	0
p_2	p_4	p_6	p_8	0	0	0	0

Rearranging and removing the zeroed columns as shown in table 8.

Table 8. Rearranged information bits

(9, 3)	(1, 3)	(2, 4)	(2, 6)
(8, 4)	(8, 6)	(9, 7)	(1, 7)
p_1	p_7	p_3	p_9
p_2	p_4	p_6	p_8

Adding the extra column in B_9 to this table will give the shortening B_9 into $\tilde{X}_5$

Table 9. $\tilde{X}_5$

(1, 9)	(9, 3)	(1, 3)	(2, 4)	(2, 6)
(2, 8)	(8, 4)	(8, 6)	(9, 7)	(1, 7)
(3, 7)	p_1	p_7	p_3	p_9
(4, 6)	p_2	p_4	p_6	p_8

Separation of $\tilde{X}_4$ will give result as shown in table 10 and 11.

Table 10. Separation of $\tilde{X}_4$ (p_1, p_7, p_3, p_9)

(9, 3)	(1, 3)	(2, 4)	(2, 6)
p_1	p_7	p_3	p_9

Table 11. Separation of $\tilde{X}_4$ (p_2, p_4, p_6, p_8)

(8, 4)	(8, 6)	(9, 7)	(1, 7)
p_2	p_4	p_6	p_8

And the separation of $\tilde{X}_5$ will give result as in table 12 and 13.

Table 12. Separation of $\tilde{X}_5$ (p_1, p_7, p_3, p_9)

(1, 9)	(9, 3)	(1, 3)	(2, 4)	(2, 6)
(3, 7)	p_1	p_7	p_3	p_9

Table 13. Separation of $\tilde{X}_5$ (p_2, p_4, p_6, p_8)

(2, 8)	(8, 4)	(8, 6)	(9, 7)	(1, 7)
(4, 6)	p_2	p_4	p_6	p_8

These give extended two B-Codes of length 5, B_5, with 4 parity bits, and the number of information bits is 6, i.e. the number of edges in K_4. In general Bohossian, Bruck, Wagner, and Xu [17] proved that the complete graph obtained from B_n (separation from $\tilde{X}_n$) can give a perfect one-factorization of K_{n+1}.

6.2. Disk Array Data Layout

In [17], perfect one-factorization is modified to obtain a *B-data layout*. B-data layout is based upon the B-code explained in the first application above. The knowledge for this constructing is that disk space is logically structured as stripe units where each stripe unit consists of a fixed number of sectors. Each stripe unit will contain either client data or redundant data. Each code symbol in B-data layout designates a stripe unit and the array code is presented via the perfect one-factorization principle.

For example let $V = \{0, 1, 2, \dots, p-1, \infty\}$ denote the vertex set. For $p > 3$, the one-factor can be obtained as:

$$P = \{(\infty, s), (s-1, s+1), \dots, (s-n+1, s+n-1)\}$$

With mod p arithmetic.

The modification of perfect one-factorization is as follows:

(1) The factors are renamed F_i for $1 \le i \le p-1$ so F_i is the factor containing (0, i). Each F_i factor loses two edges, one incident on vertex 0 and one on vertex ∞.

(2) Each edge within modified factor F_i specifies a parity value to be stored on disk drive i.

For example, consider K_7. This obviously has a perfect one-factorization.

The one-factor and the renamed factor F_i can be listed as follows:

$P_0 = \{(\infty, 0), (1, 6), (2, 5), (3, 4)\}$ $F_1 = \{(2, 6), (3, 5)\}$
$P_1 = \{(\infty, 1), (0, 2), (3, 6), (4, 5)\}$ $F_2 = \{(3, 6), (4, 5)\}$
$P_2 = \{(\infty, 2), (1, 3), (0, 4), (5, 6)\}$ $F_3 = \{(1, 2), (4, 6)\}$
$P_3 = \{(\infty, 3), (2, 4), (1, 5), (0, 6)\}$ $F_4 = \{(1, 3), (5, 6)\}$
$P_4 = \{(\infty, 4), (3, 5), (2, 6), (1, 0)\}$ $F_5 = \{(1, 4), (2, 3)\}$
$P_5 = \{(\infty, 5), (4, 6), (3, 0), (2, 1)\}$ $F_6 = \{(1, 5), (2, 4)\}$
$P_6 = \{(\infty, 6), (5, 0), (4, 1), (3, 2)\}$

Then B_6 data layouts can be obtained as shown in table 14.

Table 14. B_6

disk 1	disk 2	disk 3	disk 4	disk 5	disk 6
d_1	d_2	d_3	d_4	d_5	d_6
$d_2 \oplus d_6$	$d_3 \oplus d_6$	$d_1 \oplus d_2$	$d_1 \oplus d_3$	$d_1 \oplus d_4$	$d_1 \oplus d_5$
$d_3 \oplus d_5$	$d_4 \oplus d_5$	$d_4 \oplus d_6$	$d_5 \oplus d_6$	$d_2 \oplus d_3$	$d_2 \oplus d_4$

where $\oplus$ denotes exclusive-OR operation.

As it is explained in two examples above, perfect one-factorization is well known for its use in coding and cryptography, especially in problem related to MDS array code.

7. Conclusion

Since it was introduced in 1964, there has been a great improvement in research of proving the existence of Perfect One-Factorization for complete graph K_{2n}, for every integer $n \ge 2$. Although the exact and general result have not been accomplished yet, the studies conducted have shown that the work for this has almost reached the result. Many studies were based on various methods and basic knowledge, for example basic and abstract algebra and combinatorics. Regarding the results obtained so far, it is not impossible to conclude that the conjecture might exist. Furthermore, considering the applications, the perfect one-factorization can still be used in many fields.

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